Towards the QFT on Curved Spacetime Limit of QGR. II: A Concrete Implementation

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Abstract

The present paper is the companion of [?] in which we proposed a scheme that tries to derive the Quantum Field Theory (QFT) on Curved Spacetimes (CST) limit from background independent Quantum General Relativity (QGR). The constructions of [?] make heavy use of the notion of *semiclassical states for QGR*. In the present paper, we employ the complexifier coherent states for QGR recently proposed by Thiemann and Winkler as semiclassical states, and thus fill the general formulas obtained in [?] with life.

We demonstrate how one can, under some simplifying assumptions, explicitely compute expectation values of the operators relevant for the gravity-matter Hamiltonians of [?] in the complexifier coherent states. These expectation values give rise to effective matter Hamiltonians on the background on which the gravitational coherent state is peaked and thus induce approximate notions of $n-$particle states and matter propagation on fluctuating spacetimes. We display the details for the scalar and the electromagnetic field.

The effective theories exhibit two types of corrections as compared to the the ordinary QFT on CST. The first is due to the quantum fluctuations of the gravitational field, the second arises from the fact that background independence forces both geometry and matter to propagate on a spacetime of the form $\mathbb{R} \times \gamma$ where γ is a (random) graph.

Finally we obtain explicit numerical predictions for non-standard dispersion relations for the scalar and the electromagnetic field. They should, however, not be taken too seriously, due to the many ambiguities in our scheme, the analysis of the physical significance of which has only begun. We show however, that one can classify these ambiguities at least in broad terms.

1 Introduction

Canonical, non-perturbative Quantum General Relativity (QGR) has by now reached the status of a serious candidate for a quantum theory of the gravitational field: First of all, the formulation of the theory is mathematically rigorous. Although there are no further inputs other than the fundamental

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principles of four-dimensional, Lorentzian General Relativity and quantum theory, the theory predicts that there is a built in *fundamental discreteness* at Planck scale distances and therefore an UV cut-off precisely due to its diffeomorphism invariance (background independence). Next, while most of the results have so far been obtained using the canonical operator language, also a path integral formulation ("spin foams") is currently constructed. Furthermore, as a first physical application, a rigorous, microscopical derivation of the Bekenstein-Hawking entropy – area law has been established. The reader interested in all the technical details of QGR and its present status is referred to the exhaustive review article [?] and references therein, and to [?] for a less technical overview. For a comparison with other approaches to quantum gravity see [?, ?, ?].

A topic that has recently attracted much attention is to explore the regime of QGR where the quantized gravitational field behaves "almost classical", i.e. approximately like a given classical solution to the field equations. Only if such a regime exists, one can really claim that QGR is a viable candidate theory for quantum gravity. Consequently, efforts have been made to identify so called *semiclassical states* in the Hilbert space of QGR, states that reproduce a given classical geometry in terms of their expectation values and in which the quantum mechanical fluctuations are small [?, ?, ?, ?, ?]. Also, it has been investigated how gravitons emerge as carriers of the gravitational interaction in the semiclassical regime of the theory [?, ?, ?]. The recent investigation of Madhavan and others [?, ?, ?, ?] on the relation between the Fock representations used in conventional quantum field theories and the one in QGR further illuminate the relation between QGR and a perturbative treatment based on gravitons.

In [?] we developed and discussed a general scheme how one can define a theory of quantum matter coupled to quantum gravity in the setting of QGR and investigate its semiclassical limit. In the present paper we concretize the results of [?] by employing a specific proposal [?, ?, ?] for semiclassical states for QGR. As the present paper relies on the general approach as well as on specific results of [?], it should be read together with the latter. Especially the discussion of the conceptual issues arising in the present context is much more completely covered in [?]. Also, it should be stressed that the cautionary remarks concerning our results made there apply even more to the present paper: The analysis of the semiclassical regime of QGR in general, as well as that of the coherent states [?, ?, ?] for QGR specifically has only begun recently, and so *the main purpose of our work is exploratory.*

In the present paper, we roughly proceed in three steps: Firstly, we review the coherent state family introduced in [?, ?, ?, ?] and fix the parameters in its definition in such a way that best semiclassical behavior is obtained for the observables relevant to our considerations. Then, under some simplifying assumptions, we compute the expectation values in the coherent states for the operators relevant for setting up the effective QFT for the matter fields according to [?]. Finally, we use the resulting effective theory to approximately compute the quantum gravity corrections to the dispersion relations for the scalar and the electromagnetic field.

Let us consider these steps in more detail:

In [?, ?, ?, ?], a promising family of semiclassical states have been constructed and analyzed. Each member of this family is labelled by a (random) graph γ and a point $m \in \mathcal{M}$ in the gravitational phase space. Other states derived by the complexifier method [?] could be used as well but for the exploratory purposes of this series of papers it is sufficient to stick to those simplest ones.

Three scales enter the definition of the coherent states and are of considerable importance for their semiclassical properties. These scales are the *microscopic Planck scale* ℓ_P, the *mesoscopic graph scale* ϵ which represents the average length of an edge of γ as measured by the three metric determined by m and a *macroscopic curvature scale* L which characterizes the scale at which matter (and thus geometry) vary. While ℓ_P, L are determined by the input m, the scale ϵ is a priori a free parameter. We fix it by asking that a natural family of observables be well approximated by our coherent states which leads quite generically to a geometric mean type of behavior, concretely $\epsilon \propto \ell_P^\alpha L^{1-\alpha}$ where $0 < \alpha < \frac{1}{2}$. In contrast to the weave proposal [?] the graph scale is larger than the Planck scale due to the fact that we do not only approximate the three geometry but also the extrinsic curvature which forces the coherent state to depend on all possible spin representations of SU(2) and not only the defining (or any other single) one.

The analysis of [?] revealed that the coherent states proposed do not approximate well coordinate dependent observables like the holonomy or the electric flux operator. However, we discovered that operators which classically correspond to integrals of scalar densities of weight one are extremely well approximated. This class of observables contains Hamiltonian constraints and all spatially diffeomorphism invariant quantities which suffice to separate the points of the diffeomorphism invariant phase space. The intuitive reason for this is the following point which has been stressed for years, among others, especially by Rovelli [?, ?]: *Matter can be located only where Geometry is excited!* Classically this follows from Einstein's equations. In the quantum theory it is reflected by the fact that matter and geometry degrees of freedom are necessarily located on the same graph [?, ?]. Imagine now constructing a diffeomorphism invariant area operator $\widehat{\mathrm{Ar}}$. In contrast to its companion $\widehat{\mathrm{Ar}}(S)$ well studied in the literature it does not depend on an externally prescribed coordinate surface, rather in order to model the measurement of the are of the desk table on which you are working right now one would construct a coherent state of the combined matter and geometry Hilbert space which is peaked on flat space and, say, on an electromagnetic field which is zero everywhere except for a region in the vicinity of the table. This way the dynamics automatically forces the surface to be adapted to the graph on which the coherent state depends.

In a next step, we compute coherent state expectation values for the gravitational degrees of freedom that appear in the matter–geometry Hamiltonians. This computation, although straightforward in principle, turns out to be quite tedious in practice. To keep the computational effort on a tolerable level and maintain some clarity of presentation, we simplify things by doing the calculation only for the Abelian (Iönü-Wigner) limit $\mathrm{U}(1)^3$ of SU(2) as gauge group. The computations done in [?, ?] exemplify that this replacement does not change the results qualitatively, and therefore seems acceptable for the exploratory purposes of the present paper. A calculation in full generality should only be carried out after other issues have been settled, and will probably necessitate the use of computers.

Ground-breaking work on the phenomenology of QGR has been done in [?, ?, ?, ?]. In these works, corrections to the standard dispersion relations for matter fields due to QGR have been obtained. Since we are dealing with a theory for matter coupled to QGR in [?] and the present work, it is an important question whether these results can be confirmed in the present setting. Therefore, as a final step, we formulate effective matter Hamiltonians on a graph based on the expectation values obtained before. The resulting theory is that of fields propagation on a random graph. It bears a remarkable similarity to models considered in lattice gauge theory [?, ?, ?], and there is also a close analogy to the propagation of phonons in amorphous solids. As we have discussed at length in [?], the resulting dynamics for the matter fields is very complicated, and analytic results in the literature on lattice gauge theory and on amorphous solids are few. (To say the least. See however [?] for a beautiful numerical study of some two dimensional models from condensed matter physics.) Already a simplified one dimensional system (whose definition along with some results was sketched in [?] and will be covered more completely in [?]) shows many of the complications (optical and acoustic branches, fuzziness of dispersion relations at high energies etc.) that are to be expected for the dynamics of fields propagating in a QGR background. Therefore, to compute dispersion relations for the models obtained, we have to rely on an approximation scheme denoted "graph averages", geared to the description of the dynamics in the limit where the energy of the fields is low (or, equivalently, their wavelength large). This approximation scheme leads to precise numerical values for all correction coefficients in the dispersion relations, once we have fixed a random process that generates our sample graph. The results we obtain are similar to those of [?, ?] in many respects, but differ in the scaling of the corrections.

The validity of the approximation scheme we use has been discussed in [?] but certainly merits future investigation. In any case, the resulting formulas can probably effectively handled in full generality only by a computer.

Let us finish with a brief description of the contents of the sections to follow:

The next section contains a short review of the construction of the complexifier coherent states [?, ?, ?].

In section 3 we analyze the relation between the different scales that enter the definition of the coherent states, and their semiclassical properties. Relying on this analysis we fix the parameters of the coherent states for the rest of the paper.

Section 4 is the longest of the present paper. We show how expectation values in the coherent states can be computed and do the concrete calculations for the operators occurring in the Hamiltonians for scalar, the electromagnetic and fermionic fields coupled to gravity.

Section 5 deals with the computation of dispersion relations. We implement the procedure outlined in [?] and compare our results to the ones in the literature.

Finally, in section 6 we summarize what we have tried to do and what could be achieved with present technology. We conclude with a list of the open conceptual and technical questions that this work has left us with.

2 Complexifier Coherent States

The purpose of the present section is to review the construction and basic properties of the coherent states for QGR [?, ?, ?]. For an introduction to the formalism of QGR as a whole we refer the reader to [?, ?], or to the brief introduction in [?].

As already pointed out in the introduction, the task of constructing *semiclassical states* for QGR has received much attention [?, ?, ?, ?, ?]. Semiclassical states are states, so far in the kinematical Hilbert space of QGR, that approximate a specific classical geometry in the sense that expectation values of observables in such a state are close to the respective classical values and the quantum mechanical fluctuations are small. These requirements can certainly not be met for *all* possible observables, so the definition of a semiclassical state also involves specification of the class of observables that are well approximated.

In the present work, we will use the gauge theory coherent states (GCS for short) constructed in [?] and subsequently analyzed in [?, ?, ?]. These states are only one example of a large class of semiclassical states, called the *complexifier coherent states*. We refer to [?] for an investigation of this class of states as well as a discussion of the relationship to [?, ?, ?, ?, ?].

The main mathematical tool used in the construction of the GCS is a generalization due to Hall [?, ?] of the well known coherent states for the harmonic oscillator. The basic observation underlying this generalization is that the harmonic oscillator coherent states can be obtained as analytic continuation of the heat kernel on $\mathbb{R}^n$:

$$\psi_z^t(x) = e^{-t\Delta}\delta_{x'}(x)\Big|_{x'\longrightarrow z}, \qquad x \in \mathbb{R}^n, z \in \mathbb{C}^n,$$

the Laplacian Δ in the above formula playing the role of a *complexifier*.

It has been shown in [?] that coherent states on a connected compact Lie group G can analogously be defined as analytic continuations of the heat kernel

$$\psi_g^t(h) = e^{-t\Delta_G}\delta_{h'}^{(G)}(h)\Big|_{h'\longrightarrow u} \tag{2.1}$$

to an element u of the complexification $G^{\mathbb{C}}$ of G. These states have nice mathematical properties. Among other things, they are minimal uncertainty states for a certain pair of operators and they form an overcomplete set in the Hilbert space over G derived from the Haar measure.

The case of this construction relevant for the definition of GCS is $G = \mathrm{SU}(2)$. Its complexification is given by SL(2,$\mathbb{C}$) and can be parametrized as

$$u = \exp\left[i\tau_j p^j/2\right] h, \qquad p^k \in \mathbb{R}^3, h \in \mathrm{SU}(2) \tag{2.2}$$

where $i\tau_k, k = 1, 2, 3$ denote the Pauli matrices.

A crucial question in view of applications to the construction of semiclassical states for QGR is whether the states (2.1) obey peakedness properties analogous to that of the harmonic oscillator coherent states. In [?] it was shown that this is indeed the case: For u given by p, h via the parametrization (2.2), the following holds:

- ψ_u^t is exponentially (Gaussian) peaked with respect to the multiplication operator $\hat{h}$ on the group at the point h. The width of the peak is approximately given by $\sqrt{t}$
- ψ_u^t is Gaussian peaked with respect to the invariant vector-fields at a point p/t in the associated momentum representation. The width of the peak is approximately given by $1/\sqrt{t}$.

For a more precise formulation of these statements we refer to [?].

In QGR, the configuration degrees of freedom are represented by the holonomies along edges e of a graph γ embedded in Σ. To use the coherent states on SU(2) for the construction of semiclassical states for QGR, momentum observables, that are associated to a graph in a similar way as the holonomies have to be defined. This was done in [?]. The construction can be summarized as follows (for the many details we refer the reader to the original work): To each graph γ fix once and for all a dual 2-complex P_γ, i.e. roughly speaking a set of surfaces $(S_e)_{e\in E(\gamma)}$ which intersect each other in common boundaries at most and such that the edge e of γ intersects only S_e and that this intersection is transversal. The surfaces S_e shall be given an orientation according to the orientations of the edges e, i.e. the pairing between the orientation two form on S_e with the tangent vector field on e at the intersection point should be positive. Also to each point p lying in a surface S_e fix an analytic path $\rho(p)$ connecting the intersection point $S_e \cap e$ with p and denote the part of e from $e(0)$ to $S_e \cap e$ by e^{in}.

With the help of these structures, we can now define the quantity

$$p_j^e(A, E) = -\frac{1}{2a^2} \operatorname{Tr}\left[\tau_j h_{e^{\text{in}}} \left(\int_{S_e} h_{\rho(p)} E^a(p) h_{\rho(p)}^{-1} \epsilon_{abc}\, dS^{bc}(p)\right) h_{e^{\text{in}}}^{-1}\right]. \tag{2.3}$$

where a is a length scale introduced to make p_j^e dimensionless and whose relation with t is $t = \ell_P^2/a^2$. The key feature of this new variable p_j^e is that

$$\left\{p_j^e, h_{e'}\right\} = \frac{\kappa^2}{a^2}\delta_{e,e'}\frac{\tau_j}{2}h_{e'}, \qquad \left\{p_i^e, p_j^{e'}\right\} = -\frac{\kappa^2}{a^2}\delta_{e,e'}\epsilon_{ijk}p_k^e \tag{2.4}$$

where ϵ_{ijk} are the structure constants of SU(2). Therefore, if h_e is represented by the multiplication operator $\widehat{h}$ on the cylindrical subspace corresponding to e, p_j^e can be represented by the right invariant vector-field itX^j acting on the cylindrical subspace corresponding to e.

Having the momentum variables p_j^e at disposal, the construction of the GCS can now be finished. It needs three inputs:

- A point $(A^{(0)}, E^{(0)})$ in the classical phase space that should be approximated.
- A graph γ and a corresponding dual polyhedronal decomposition P_γ of Σ, and the associated path system Π_γ.
- The parameter t or, equivalently, the length scale a.

For each edge e of the graph γ, one can now compute the holonomy $h_e^{(0)}$ in the classical connection $A^{(0)}$ and the classical quantities $p_j^{(0)e}$ depending on $A^{(0)}, E^{(0)}$ as expressed in (2.3). The gauge coherent state for QGR is then defined as

$$\psi^t_{(A^{(0)},E^{(0)})}(h_{e_1}, \ldots, h_{e_N}) \doteq \prod_{n=1}^{N} \psi^t_{g_{e_n}(A_0,E_0)}(h_{e_n})$$

where $e_1, \ldots, e_N$ represent the edges of the graph γ and $g_e(A_0, E_0) = \exp(p_j^{(0)e}\tau_j/2)h_e^{(0)}$.

The states thus defined inherit the peakedness properties of the coherent states (2.1) in an obvious way with respect to the elementary observables $\widehat{h}_{e_1}, \ldots, \widehat{h}_{e_N}$ and $\widehat{p}_j^{e_1}, \ldots, \widehat{p}_j^{e_N}$. For more complicated observables, a more detailed consideration has to be given. This is the topic of the next section. We will see that this analysis fixes the parameter t as well as the average edge length of the graph G, thus reducing the freedom in the construction of the GCS considerably.

3 Observables and Scales

In the previous section we saw that the complexifier coherent states $\psi_{m,\gamma}$ that will be used in this paper (see [**?**] for generalizations) depend on a point $m \in \mathcal{M}$ and a triple $(\gamma, P_\gamma, \Pi_\gamma)$ where γ is a graph, P_γ a polyhedronal decomposition of Σ dual to γ and Π_γ is an associated path system. The states $\pi_{m,\gamma}$ are linear combinations of spin network states over γ (and all of its subgraphs) with coefficients which depend on m, P_γ, Π_γ. We are interested in the question which kind of operators $\hat{O}$ are approximated well by these states, that is, for which holds that expectation values are close to the classical value and for which the fluctuations are small.

By construction, they approximate very well the holonomy operators $\hat{h}_e$ and the electric flux operators $\hat{E}_j(S_e)$ where e runs through the set of edges of γ and S_e is a face in the polyhedronal decomposition dual to e. But how about more general operators such as $\hat{h}_p, \widehat{\mathrm{Ar}(S)}$ where p is an arbitrary path and S an arbitrary surface? First of all, unless p is a composition of edges of γ we have $<\psi_{m,\gamma}, \hat{h}_e \psi_{m,\gamma}>= 0$ due to the orthogonality of spin-network states. Secondly, the expectation values of the area operator suffer from the "staircase problem" [?] which says that unless S is composed of the S_e then its expectation value will be off the correct value.

The first reaction is: The states are not good, they must be improved. One such improvement could be by averaging over an ensemble of graphs [?] but as shown in [?] this still does not improve the holonomy expectation values. Thus, one could think that one should construct semiclassical states of a completely different type, maybe going to a new representation of the canonical commutation relations [?, ?, ?]. However, this is not easy if the present formulation of QGR is to be kept as shown in [?]. It therfore seems that we are in trouble.

There is a second possibility however: Maybe we are just trying to approximate the wrong observables? Notice that it is a *physical input* which observables should be well approximated, certainly we do not expect all classical quantities to be approximated well in the quantum theory. This is even true for simple finite dimensional systems such as the harmonic oscillator: The energy itself is well approximated but not its exponential. In our case, traces of holonomy operators and area operators are certainly natural candidates for operators to be well approximated because they are gauge invariant, suffice to separate the points of the gravitational phase space and are simple functions of the basic operators that the whole quantization is based on, namely holonomy and electric flux operators. Is it possible that there are observables which are better suited for our semiclassical considerations?

A first hint of how such observables should look like comes from the observation that the volume operator $\widehat{\mathrm{Vol}}(R)$ for a coordinate region R does not suffer from the staircase problem. A detailed analysis shows that this happens because the region R corresponds to a *three-dimensional submanifold* of Σ rather than one – or two dimensional ones. We therefore are led to the proposal that one should not look at holonomy and area operators but rather at quantities that classically come from three-dimensional integrals. There are classical observables of that kind that one can construct and which separate the points of the gauge invariant gravitational phase space as well: Let ω_a be a one form, say of rapid decrease, and consider

$$
\begin{aligned}
Q(\omega) &:= \int_\Sigma d^3x \frac{E^a_j E^b_j}{\sqrt{\det(q)}} \omega_a \omega_b && (3.1) \\
M(\omega) &:= \int_\Sigma d^3x \frac{B^a_j B^b_j}{\sqrt{\det(q)}} \omega_a \omega_b && (3.2)
\end{aligned}
$$

where $E^a_j E^b_j =: \det(q) q^{ab}$ and $B^a_j = \frac{1}{2}\epsilon^{abc} F^j_{bc}$ and where F is the curvature of the connection A. Notice that both (3.1), (3.2) are of the type of operators that can be quantized with the methods of [?] in a background independent fashion since they are integrals of scalar densities. Moreover, they suffice to separate the points of the gauge invariant phase space as one can see by suitably restricting the support of ω_a and by the polarization identity for quadratic forms.

The crucial fact about these quantities is now as follows: When we quantize them along the lines of [?] they become diffeomorphism covariant, densely defined, closed operators on the kinematical QGR Hilbert space $\mathcal{H}_0$ of the following structure

$$\hat{O}(\omega)T_s = \sum_{v\in V(s(\gamma))} \sum_{v\in\partial e,\partial e';e,e'\in E(\gamma(s))} \omega(e)\omega(e')\hat{O}_{v;e,e'}T_s =: \sum_{v\in V(s(\gamma))} \hat{O}_\gamma(\omega,v)T_s \tag{3.3}$$

where T_s is a spin-network state with underlying graph $\gamma(s)$ and $V(\gamma), E(\gamma)$ denote the sets of vertices and edges of a graph respectively and $\omega(e) = \int_e \omega$. The fact that an action only at vertices takes place in (3.3) is due to the appearance of the volume operator which enters the stage due to the factor of $1/\sqrt{\det(q)}$ in (3.1), (3.2) which is required by background independence and the requirement that only density one valued quantities can be quantized in a background independent way [?]. The operator $\hat{O}_{v;e,e'}$ is a polynomial formed out of holonomy operators *along the edges of* $\gamma(s)$ and powers of the volume operator restricted to an arbitrarily small neighborhood of the vertex v. Now the coherent states are constructed precisely in such a way that the holonomy operators along the edges of $\gamma(s)$ are well approximated and, as we will explicitly prove in this work, they also approximate very well the volume operator of [?, ?] *at least if the graph is six-valent, e.g. of cubic topology.* (For other graph topologies the prefactor $\frac{1}{8\cdot 3!}$, which enters the square roots that defines the volume operator, would presumably need to be adapted to the vertex valence, it should be larger (smaller) for valences smaller (larger) than six).

Thus, due to the Ehrenfest properties proved in [?] we conclude that at least for coherent states based on graphs with cubic topologies the operators (3.1), (3.2) are approximated well (with small fluctuations) provided the expectation values of (3.3) define a Riemann sum approximation of the classical integrals (3.1), (3.2). This is, however, the case by the very construction of such operators as outlined in [?]. Thus, the mechanism responsible for the fact that no such problems as for the area and holonomy operators arise is due to the fact that for operators coming from volume integrals the elementary electric flux and holonomy operators involved are automatically those adapted to the graph in question.

That only cubic graphs should give rise to the correct classical limit might be disturbing at first but it is on the other hand not too surprising: The volume operator at a given vertex v is a square root of an operator which in turn is a sum of basic operators, one for each unordered triple of distinct edges incident at v in [?] and in [?] one considers only those triples which have linearly independent tangents at v. Each of these basic operators is a third order homogeneous polynomial in electric flux operators. With respect to our coherent states, each polynomial gives a contribution of the same order of magnitude. If n is the valence of the vertex of v then there will be altogether $N(n)$ terms where $N(n) = n[n-1][n-2]$ for the operator in [?] while for the operator of [?] this number is smaller whenever there are triples of edges with co-planar tangents at v. The smallest valence for which the volume operator does not vanish is $n = 3$ in which case $N(3) \leq 6$. Since each term corresponds to the volume of the cell of the polyhedronal decomposition dual to γ, the factor $1/48$ dividing the sum over triples is too large. Now $N(4) \leq 24$ is still too small while $N(5) \leq 120$ is already definitely too large for the volume operator of [?]. For the cubic topology we have, however, precisely $N(6) = 48$ for [?] because the only triples that contribute are formed by those spanning the eight octants defined by the coordinate system defined by the tangents of the six edges at v. For graphs of higher valence, unless there are sufficiently many coplanar triples, the [?] volume operator also over-counts the classical volume. Notice that none of these statements proves that one operator is proved over the other, it just means that our coherent states do not approximate both equally well. Only if one would know that our states are "the correct choice", could one distinguish between the two kinds of volume operators on physical grounds.

Intuitively, it is actually not too bad that only graphs of low valence should give rise to the correct classical limit. After all, one would not try to approximate a classical integral by Riemann sums in terms of graphs with vertices of arbitrarily high topology. Such graphs should describe quantum states without classical correspondence. It is also natural that cubic graphs are somehow distinguished because the classical integral is locally defined by a Cartesian coordinate system.

Having convinced ourselves that the coherent states of the previous section actually do make sense at least for operators of the kind (3.1) and (3.2) we turn to the question how the scale ϵ should be chosen. In order to quantize the classical integral

$$O(\omega) = \int_\Sigma d^3x O^{ab}\omega_a\omega_b \tag{3.4}$$

the procedure adopted in [?] was to define the operator on the spin-network basis. Thus, let γ be a graph and $v \mapsto R_v$ a partition of Σ where v runs through $V(\gamma)$. Let ϵ_v^3 be the coordinate volume of R_v. Then we have

$$O(\omega) = \sum_{v\in V(\gamma)} \int_{R_v} d^3x O^{ab}\omega_a\omega_b \approx \sum_v \epsilon_v^3 O^{ab}(v)\omega_a(v)\omega_b(v) =: O_\gamma(\omega) \tag{3.5}$$

where in the last step we have replaced the integral by a Riemann sum. The quantization of the term at v in the sum in (3.5) gives rise to the operator

$$\hat{O}_\gamma(\omega, v) = \sum_{v\in\partial e,\partial e';e,e'\in E(\gamma(s))} \omega(e)\omega(e')\hat{O}_{v;e,e'} \tag{3.6}$$

in (3.3) and by construction its expectation value in a coherent state $\psi_{\gamma,m}$ gives back $O_\gamma(\omega, v)_{|m}$ to zeroth order in $\hbar$. Thus, apart from quantum corrections for the expectation value, which we will call a *normal ordering error*, already the quantity $O(\omega, m) - O_\gamma(\omega_m)$ is in general non-zero. This *classical error* error will decrease with ϵ. With the Euler-MacLaurin error estimation methods of [?] one can prove an estimate of the form

$$|O(\omega, m) - O_\gamma(\omega, m)| \leq [\frac{\epsilon}{L}]^\beta O(\omega, m) \tag{3.7}$$

where $\beta \geq 2$ and L is the average size of the the quantity $[O^{ab}\omega_a\omega_b]''/[O^{ab}\omega_a\omega_b]$ where the double prime denotes second derivatives. Thus, L captures information about the gravitational curvature as well as the curvature of ω. The size of β depends strongly on the randomness of the graph in question and also would change if one would average over graphs.

More precisely, if we are interested in diffeomorphism invariant quantities (3.1), (3.2) such as the matter Hamiltonians that we wish to approximate in the following sections, then we should set, e.g., $\omega_a = \phi_{,a}$ where ϕ is a scalar field or we should consider integrands of the form $q_{ab}E^aE^b/\sqrt{\det(q)}$ where E^a is the Maxwell electric field. To see what the matter and geometry scales involved are, consider the time – time component of the Einstein equations for electromagnetic waves with vector potential $A = A_0 e^{i(|k|t-kx)}$. If q^2 is the electric charge, then the matter energy density is of the order $A_0^2 k^2/q^2$. If R denotes the curvature radius of the curvature tensor then we get from Einstein's equations $R^{-2} \approx (\ell_P A_0 k)^2/\alpha$ where $\alpha = \hbar q^2$ is the Feinstruktur constant. Thus, if we introduce the wave length by $k = 1/\lambda$ then $R^{-2} \approx (10 A_0 \ell_P)^2 \lambda^{-2} \ll \lambda^{-2}$ at least for weak electromagnetic waves $A_0 \ll 10^{32}\mathrm{cm}^{-1}$. Thus, L should, for the applications of this paper to be thought of being very close to the matter wave length λ and R is large, so that the geometry is almost flat.

Let us now consider fluctuations. Since the quantities $M(\omega), Q(\omega)$ have different physical units, in order to compare their fluctuations we should compare their relative fluctuations which are dimension-free quantities. More precisely, we consider the expectation value in the coherent state $\psi_{\gamma,m}$ of the relative deviation squared $[\hat{O}/O(m) - 1]^2$ between the operator $\hat{O}$ and its expected classical value $O(m)$ which is a proper measure for the total deviation of the operator from the classical quantity due to 1) the fluctuation of the gravitational field and 2) its discrete nature which forces us to work with graphs rather than continuous integrals. If we denote by $< . >_{m,\gamma}$ the expectation value in the coherent state $\psi_{m,\gamma}$ and if there is no normal ordering error then we arrive at

$$< [\frac{\hat{O}}{O(m)} - 1]^2 >_{m,\gamma} \approx < [\frac{\hat{O}_\gamma}{O_\gamma(m)} - 1]^2 >_{m,\gamma} + [\frac{\hat{O}_\gamma(m)}{O(m)} - 1]^2 \tag{3.8}$$

The second term in (3.8) is of order $(\epsilon/L)^{2\beta}$ as derived above. Now we see that for the quantity $M(\omega)$ the first term is divergent for flat data because $M(\omega) = 0$ while for $O = Q(\omega)$ there is no such problem. This is like comparing the relative fluctuations of $\hat{x}, \hat{p}$ for the harmonic oscillator at the phase space point $(x,p) = (0,1)$ which of course makes little sense. To deal with this problem we chose the following strategy: We compare the relative fluctuations at *generic points in phase space* where we find a relation between the scale a of the coherent state and the scale L and *then extend this relation to all points in* $\mathcal{M}$. This strategy is certainly ad hoc but we do not see any other possibility at this point to fix the size of a by a more physical requirement.

Accepting this we will consider non-flat data in which case generically $L \approx R$ is closer to the curvature scale. If we assume that the operators $\hat{O}_\gamma(\omega, v)$ in (3.3) are much weaker correlated for distinct v than for coinciding v (as it turns out to be the case) then we obtain

$$< \hat{O}_\gamma^2 >_{m,\gamma} - (< \hat{O}_\gamma >_{m,\gamma})^2 \approx \sum_v [< \hat{O}_{v,\gamma}^2 >_{m,\gamma} - (< \hat{O}_{v,\gamma} >_{m,\gamma})^2] \tag{3.9}$$

Restricted to γ, the operator $\hat{O}_{v,\gamma}$ is a homogeneous polynomial of some rational power of the operators $P^e_j \approx E_j(S_e)/a^2$ which are of order $E_0\epsilon^2/a^2$ where E_0 is some average value of E^a_j and a is the coherent state scale introduced in the previous section. It is also a polynomial of some integral power of the operator $h_\alpha - h_\alpha^{-1}$ which is of order $B_0\epsilon^2$ where B_0 is some average value of B^a_j and is approximately given by $E_0 L^{-2}$. As shown in [**?**], the fluctuations for the respective vertices v is effectively given by exchanging $O_{\gamma,v}(m)$ by $t\partial^2 O_{\gamma,v}(m)/[\partial P(S_e)]^2 \approx t O_{\gamma,v}/P(S_e)^2$ for the electric fluctuations and by $t\partial^2 h_\alpha^2 O_{\gamma,v}(m)/[\partial h_\alpha]^2 \approx t O_{\gamma,v}/[h_\alpha - h_\alpha^{-1}]^2$ for the magnetic ones where α is some loop incident at v. Inserting $P(S_e) \approx E_0\epsilon^2/a^2$ and $h_\alpha - h_\alpha \approx E_0\epsilon^2/R^2 \approx E_0\epsilon^2/L^2$. equating (3.9) for $O = M(\omega), O = Q(\omega)$ respectively immediately leads to $a \approx L$.

While the derivation of this result is maybe not entirely convincing, it is actually the only choice from a classical point of view: Since L is the only classical scale available and the complexifier generator C for our coherent states, from which the scale a derives, is a classical object, the scale L is the only classical one in the problem that should be used in order to make $C/\hbar$ dimensionfree.

Coming back to flat space $m = (A_a^j, E_j^a) = (0, \delta_j^a)$ we want to fix ϵ by requiring that the relative fluctuation (3.8) for $O = Q(\omega)$ is minimized. This leads to the condition that (notice $E_0 = 1$)

$$t \frac{\epsilon^3}{\mathrm{Vol(supp}(\omega))} \frac{1}{[\epsilon^2/a^2]^2} + [\epsilon^2/L^2]^\beta \tag{3.10}$$

be minimized where $a := L$. The fluctuation contribution depends on the volume of the support of ω. Since we want to resolve regions with our graph of the linear size bigger or equal than L (think of L as the smallest wavelength to be resolved for our applications) we obtain that (3.10) is certainly dominated by

$$t \frac{L}{\epsilon} + [\epsilon^2/L^2]^\beta \tag{3.11}$$

This function has a unique minimum at

$$\epsilon = \ell_P^\alpha L^{1-\alpha}, \quad \alpha = \frac{1}{\beta + \frac{1}{2}} \le \frac{2}{5} < \frac{1}{2} \tag{3.12}$$

In [?] we chose $\mathrm{Vol(supp}(\omega)) \ge \epsilon^3$ and different observables, adapted to the graph in question, in order to have the lattice degrees of freedom well approximated and led to $\alpha \approx 1/6$. However, it is clear that this choice would lead to boundary effects if the support of ω is not adapted to the graph in question which would be unnatural. Such boundary effects are avoided by $\mathrm{Vol(supp}(\omega)) \ge L^3$ and go at most as the quotient between the volume of a shell of thickness ϵ at the boundary of a region of volume L^3 and its volume, that is, as ϵ/L. This drives the lattice scale ϵ closer to the Planck scale. Notice that in any case $\ell_P \ll \epsilon \ll L$.

This concludes the present section. The relations $a := L$ and (3.12) will be our working proposal.

4 Coherent States Expectation Values

The purpose of this section is to present the calculation of the expectation values of the various terms occurring in the Hamiltonians of section 4 in [?] in the coherent states for QGR discussed in the preceeding sections. In the first part we will explain the simplifying assumptions used for the computation and introduce the necessary notation. Section 4.2 is devoted to the computation of the expectation values of the volume operator $\widehat{V}_v$ and the operator

$$\hat{Q}_e^j(v, r) = \frac{1}{4r} \mathrm{tr}(\tau_j h_e [h_e^{-1}, (\hat{V}_v)^r]), \tag{4.1}$$

as they are the basic building blocks of the Hamiltonians obtained in [?]. In section 4.3 the results are used to give the expectation values of the geometric operators occurring in the Hamiltonians for the scalar and the electromagnetic field.

4.1 Implementation of the Simplifying Assumptions

The cubic lattice:

For reasons already explained in our companion paper, the first simplification that we will make concerns the random graphs: In the following we will exclusively work with states based on graphs of cubic topology. This simplifies both the notation and the c-number coefficients occurring in the Hamiltonians. In a graph of cubic topology, each vertex is six-valent with three edges ingoing and three outgoing. We denote the outgoing edges by e_I, $I=1,2,3$ and choose an ordering, such that the tangents of e_1, e_2, e_3 form a right handed triple wrt. the given orientation of Σ. The vertices can be labeled by elements v of $\mathbb{Z}^3$. We write $e_I^+(v) := e_I(v),\ e_I^-(v) := e_I(v-I)^{-1}$ where $n-I$ denotes the point in $\mathbb{Z}^3$ translated one unit along the negative I axis. In keeping with that convention, we associate to $e_I^-(v)$ the dual surface $S_{e_I(v-I)}$ with its orientation *reversed*.

Replacing $\mathrm{SU}(2)$ **by** $\mathrm{U}(1)^3$**:**
We substitute $SU(2)$ by $U(1)^3$ in our computation because the results of [?, ?] reveal that the qualitative features are untouched so nothing conceptually new is learned when doing the much harder non-Abelian computation. For the exploratory purposes of this paper it is thus sufficient to stick with the Abelian group. Consequently we will replace $\widehat{Q}$ as well as the volume operator itself by appropriate $\mathrm{U}(1)^3$ counterparts. For $\mathrm{U}(1)^3$ each edge is not labelled by a single, non-negative, half-integral spin degree of freedom but rather by three integers $n_j \in \mathbb{Z},\ j=1,2,3$ and we have three kinds of holonomies h_e^j. The generators τ_j of $\mathrm{U}(1)^3$ are simply i (imaginary unit). The canonical commutation relations on $L^2(\mathrm{U}(1)^3, d^3\mu_H)$ are replaced by

$$
\begin{aligned}
[\widehat{h}^j, \widehat{h}^k] &= 0 \\
[\widehat{p}_j, \widehat{h}^k] &= it\delta_j^k \widehat{h}^j \\
[\widehat{p}_j, \widehat{p}_k] &= 0
\end{aligned}
$$

(cf. (2.4)) with adjointness relations $(\widehat{h}^j)^\dagger = (\widehat{h}^j)^{-1}$, $(\widehat{p}_j)^\dagger = \widehat{p}_j$. It follows that (4.1) gets replaced by

$$
\widehat{Q}_e^j(v,r) = \frac{i}{4r}\widehat{h}_e^j\left[(\widehat{h}_e^j)^{-1}, \widehat{V}_v^r\right],
$$

Finally the expression for the volume operator in our companion paper is replaced by

$$
\widehat{V}_{\gamma,v} = l_p^3 \sqrt{|\epsilon^{jkl}[\frac{\widehat{Y}_j^{e_1^+(v)} - \widehat{Y}_j^{e_1^-(v)}}{2}]\,[\frac{\widehat{Y}_k^{e_2^+(v)} - \widehat{Y}_k^{e_2^-(v)}}{2}]\,[\frac{\widehat{Y}_l^{e_3^+(v)} - \widehat{Y}_l^{e_3^-(v)}}{2}]|}
$$

with $\widehat{Y}_j^e = ih^j\partial/\partial h^j$.

The $\mathrm{U}(1)^3$ coherent states over any graph γ are given by (see [?])

$$
\psi_{\gamma,m}^t = \otimes_{e\in E(\gamma)} \otimes_{j=1}^3 \psi_{g_e^j(m)}^t
$$

where

$$
\psi_g^t = \sum_{n\in\mathbb{Z}} e^{-tn^2/2}(gh^{-1})^n
$$

and $g_e^j(m) = e^{p_j^e(m)}h_e^j(m) \in \mathbb{C}-\{0\} = \mathrm{U}(1)^{\mathbb{C}}$. Here m is a point in the gravitational phase space and

$$
h_e^j(m) \doteq \mathcal{P}\exp(i\int_e A^j)
$$

$$p_j^e(m) \doteq \frac{1}{a^2}\int_{S_e}(*E)_j$$

that is, due to the Abelian nature of our simplified gauge group the path system in S_e is no longer needed.

As is obvious from the explicit form of the Hamiltonians, our calculation can be done vertex by vertex since there is no inter-gravitational interaction between the associated operators. We can therefore concentrate on a single vertex for the remainder of this section and drop the label v in what follows.

For the sake of the computation to follow, we introduce the shorthands

$$h_{J\sigma j} \doteq h^j_{e^\sigma_J}, \qquad p_{J\sigma j} \doteq p_j^{e^\sigma_J}, \qquad g_{J\sigma j} \doteq e^{p_{J\sigma j}} h_{J\sigma j}$$

and similarly the operators $\widehat{Y}_{J\sigma j} \doteq \widehat{Y}_j^{e^\sigma_J}$. Let us finally define

$$\widehat{\Box} \doteq \frac{1}{a^3}\widehat{V}, \qquad \widehat{q}_{J\sigma j}(r) \doteq \frac{r}{2ita^{3r}}\widehat{Q}^j_{e^\sigma_J}(r). \tag{4.2}$$

Note that $\widehat{q}$ is essentially selfadjoint.

The huge advantage of $\mathrm{U}(1)^3$ over SU(2) is that the "spin-network functions"

$$T_{\{n_{J\sigma j}\}}(\{h_{J\sigma j}\}) = \prod_{J\sigma j} h_{J\sigma j}^{-n_{J\sigma j}}$$

are simultaneous eigenfunctions of all the $\widehat{Y}_{J\sigma j}$ with respective eigenvalue $n_{J\sigma j}$. Even better, the operator $\widehat{q}_{J_0\sigma_0 j_0}(r)$ is also diagonal with eigenvalue

$$\lambda^r_{J_0\sigma_0 j_0}(\{n_{J\sigma j}\}) = 2\frac{\lambda^r(\{n_{J\sigma j}\}) - \lambda^r(\{n_{J\sigma j} + \delta_{(J_0\sigma_0 j_0),(J\sigma j)}\})}{t},$$

where

$$\lambda^r(\{n_{J\sigma j}\}) = t^{3r/2}\left(\sqrt{|\epsilon^{jkl}[\frac{n_{1,+,j}-n_{1,-,j}}{2}]\,[\frac{n_{2,+,k}-n_{2,-,k}}{2}]\,[\frac{n_{3,+,l}-n_{3,-,l}}{2}]|}\right)^r.$$

4.2 The Expectation Values of $\widehat{q}$

Now we will explicitly calculate the expectation values of powers of the operators $\widehat{q}$ and $\widehat{\Box}$. The gravitational parts of the matter Hamiltonians constructed in [**?**] are all sums and products of these operators which act only on the edges of a specific vertex, therefore we can restrict consideration to a single vertex and consequently to a part

$$\psi^t_{\{g_{J\sigma j}\}}(\{h_{J\sigma j}\}) \doteq \prod_{J\sigma j}\psi^t_{g_{J\sigma j}}(h_{J\sigma j})$$

of the coherent state which just contains the factors corresponding to the edges of a single vertex. What we are looking for is the expectation value of an arbitrary polynomial of the $\widehat{q}$:

$$\langle\,\cdot\,\rangle \doteq \frac{\langle\psi^t_{\{g_{J\sigma j}\}}, \prod_{k=1}^N \widehat{q}_{J_k\sigma_k j_k}(r_k)\psi^t_{\{g_{J\sigma j}\}}\rangle}{||\psi^t_{\{g_{J\sigma j}\}}||^2}$$

$$= \frac{\sum_{\{n_{J\sigma j}\}} e^{-t\sum_{J,\sigma,j} n_{J\sigma j}^2} e^{2\sum_{J\sigma j} p_{J\sigma j} n_{J\sigma j}} \prod_{k=1}^N \lambda^r_{J_k\sigma_k j_k}(\{n_{J\sigma j}\})}{\prod_{J,\sigma,j} ||\psi^t_{g_{J\sigma j}}||^2} \tag{4.3}$$

where (see [?])

$$||\psi^t_g||^2 = \sqrt{\frac{\pi}{t}} e^{p^2/t}[1+K_t(p)],\ g = e^p e^{i\varphi},\ |K_t(p)| \le K_t = O(t^\infty). \tag{4.4}$$

As in [?], in order to extract useful information out of the formula (4.3) it is of outmost importance to perform a Poisson transformation on it because we are interested in tiny values of t for which (4.3) converges rather slowly while the transformed series converges rapidly since then t gets replaced by $1/t$. To that end, let us introduce $T \doteq \sqrt{t}, x_{J\sigma j} \doteq T n_{J\sigma j}$, whereupon

$$\langle\cdot\rangle = \frac{\sum_{\{x_{J\sigma j}\}} e^{-\sum_{J,\sigma,j} x_{J\sigma j}^2} e^{2\sum_{J\sigma j} x_{J\sigma j} p_{J\sigma j}/T} \prod_{k=1}^N \lambda^r_{J_k\sigma_k j_k}(\{x_{J\sigma j}\})}{\prod_{J,\sigma,j} ||\psi^t_{g_{J\sigma j}}||^2} \tag{4.5}$$

where

$$\begin{aligned}
\lambda^r_{J_0\sigma_0 j_0}(\{x_{J\sigma j}\}) &= 2\frac{\lambda^r(\{x_{J\sigma j}\}) - \lambda^r(\{x_{J\sigma j} + T\delta_{(J_0\sigma_0 j_0),(J\sigma j)}\})}{t} \\
\lambda^r(\{x_{J\sigma j}\}) &= t^{3r/4}\sqrt{|\epsilon^{jkl}[\frac{x_{1,+,j}-x_{1,-,j}}{2}]\,[\frac{x_{2,+,k}-x_{2,-,k}}{2}]\,[\frac{x_{3,+,l}-x_{3,-,l}}{2}]|}^{\,r}
\end{aligned} \tag{4.6}$$

Then Poisson's theorem gives

$$\langle\cdot\rangle = \frac{\frac{1}{T^{18}}\sum_{\{n_{J\sigma j}\}}\int_{\mathbb{R}^{18}} d^{18}x e^{\sum_{J,\sigma,j}[-x_{J\sigma j}^2 + 2x_{J\sigma j}(p_{J\sigma j} - i\pi n_{J\sigma j})/T]} \prod_{k=1}^N \lambda^r_{J_k\sigma_k j_k}(\{x_{J\sigma j}\})}{\prod_{J,\sigma,j} ||\psi^t_{g_{J\sigma j}}||^2} \tag{4.7}$$

An observation that reduces the eighteen dimensional integral to a nine dimensional one is that the integrand in (4.7) only depends on $x_{Jj} \doteq x^-_{Jj} \doteq [x_{J,+,j} - x_{J,-,j}]/2$ and not on $x^+_{Jj} \doteq [x_{J,+,j} + x_{J,-,j}]/2$. Consider also the analogous quantities $p^\pm_{Jj} \doteq [p_{J,+,j} \pm p_{J,-,j}]/2,\ n^\pm_{Jj} \doteq [n_{J,+,j} \pm n_{J,-,j}]/2$ and let $p_{Jm} \doteq p^-_{Jj},\ n_{Jm} \doteq p^-_{Jj}$. Switching to the coordinates $x^\pm_{Jj}$, noticing that $|\det(\partial\{x_{J\sigma j}\}/\partial\{x^+_{Jj}, x^-_{Jj}\}| = 2^9$ we obtain

$$\begin{aligned}
\langle\cdot\rangle = &\frac{(\frac{2}{t})^9 \sum_{\{n_{J\sigma j}\}}[\int_{\mathbb{R}^9} d^9x^+ e^{2\sum_{Jj}[-(x^+_{Jj})^2 + 2x^+_{Jj}(p^+_{Jj} - i\pi n^+_{Jj})/T]}]}{\prod_{J,\sigma,j} ||\psi^t_{g_{J\sigma j}}||^2} \times \\
&\times \left[\int_{\mathbb{R}^9} d^9x e^{2\sum_{Jj}[-x^2_{Jj} + 2x_{Jj}(p_{Jj} - i\pi n_{Jj})/T]} \prod_{k=1}^N \lambda^r_{J_k\sigma_k j_k}(\{x_{Jj}\})\right]
\end{aligned} \tag{4.8}$$

where

$$\begin{aligned}
\lambda^r_{J_0\sigma_0 j_0}(\{x_{Jj}\}) =& 2\frac{\lambda^r(\{x_{Jj}\}) - \lambda^r(\{x_{Jj} + T\delta_{(J_0 j_0),(Jj)}/2\})}{t} =: \lambda^r_{J_0 j_0}(\{x_{Jj}\}) \\
\lambda^r(\{x_{Jj}\}) &= t^{3r/4}\,(|\det(\{x_{Jj}\})^{r/2}
\end{aligned} \tag{4.9}$$

actually *no longer depends on* σ_0! The integral over x^+_{Jj} in (4.9) can be immediately performed by using a contour argument with the result

$$\langle\,\cdot\,\rangle = \frac{(\frac{\sqrt{2\pi}}{t})^9 \sum_{\{n_{J\sigma j}\}} e^{\frac{2}{t}\sum_{Jj}(p^+_{Jj}-in^+_{Jj})^2} [\int_{\mathbb{R}^9} d^9x e^{2\sum_{Jj}[-x^2_{Jj}+2x_{Jj}(p_{Jj}-i\pi n_{Jj})/T]} \prod_{k=1}^N \lambda^r_{J_k\sigma_k j_k}(\{x_{Jj}\})]}{\prod_{J,\sigma,j} ||\psi^t_{g_{J\sigma j}}||^2} \tag{4.10}$$

Finally, using (4.4) we can further simplify to

$$\langle\,\cdot\,\rangle = \frac{\sqrt{\frac{2}{\pi}}^9}{[(1-K_t)^{18},(1+K_t)^{18}]} \sum_{\{n_{J\sigma j}\}} e^{\frac{2}{t}\sum_{Jj}[(p^+_{Jj}-i\pi n^+_{Jj})^2-(p^+)^2_{Jj}-p^2_{Jj}]} \times$$
$$\times \int_{\mathbb{R}^9} d^9x e^{2\sum_{Jj}[-x^2_{Jj}+2x_{Jj}(p_{Jj}-i\pi n_{Jj})/T]} \prod_{k=1}^N \lambda^r_{J_k\sigma_k j_k}(\{x_{Jj}\}) \tag{4.11}$$

where the notation for the denominator means that its value ranges at most in the interval indicated. Its precise value will be irrelevant for what follows since its departure from unity is $O(\infty)$.

Only the $n_{J\sigma,j}=0$ terms matter:

The remaining integral in (4.11) cannot be computed in closed form so that we must confine ourselves to a judicious estimate. We wish to show that the only term in the infinite sum of (4.11) which contributes corrections to the classical result of finite order in t is the one with $n_{J\sigma j}=0$ for all J,σ,j. In order to do that, we must demonstrate that all the other terms can be estimated in such a way that the series of their estimates converges to an $O(t^\infty)$ number. This would be easy if we could complete the square in the exponent of the integrand but since for $r/2$ not being an even positive integer the function λ^r is not analytic in $\mathbb{C}^9$ we cannot immediately use a contour argument in order to estimate the remaining integral. In order to proceed and to complete the square anyway we expand the product $\prod_{k=1}^N \lambda^r_{J_k\sigma_k j_k}(\{x_{Jj}\})$ into monomials of the form $\prod_{k=1}^N \frac{\lambda^r(\{x_{Jj}+c^k_{Jj}\})}{t}$ with $c^k_{Jj}=T\delta_{J_kj_k,Jj}/2$ or $c^k_{Jj}=0$ and estimate the integrals over the latter. We trivially have

$$\lambda^r(\{x_{Jj}+c^k_{Jj}\}) = t^{3r/4}([\det(\{x_{Jj}+c^k_{Jj}\})]^2)^{r/4} = t^{3r/4}\exp\left(\frac{r}{4}\ln([\det(\{x_{Jj}+c^k_{Jj}\})]^2)\right) \tag{4.12}$$

where we must use the branch of the logarithm with $\ln(z)=\ln(|z|)+i\varphi$ for any complex number $z=|z|e^{i\varphi}$ with $\varphi\in[0,2\pi)$. With this branch understood, in the form (4.12) the integrand of (4.11) becomes univalent on the entire complex manifold $\mathbb{C}^9$ except at the points where $\det(\{x_{Jj}+c^k_{Jj}\})=0$. Now a laborious contour argument can be given tho the extent that we can move the path of integration away from the real hyperplane in $\mathbb{C}^9$ without changing the result. Therefore we can indeed complete the square in the exponent.

It remains to estimate (4.11) from above. Isolating the term with $n_{J\sigma,j}=0$ for all J,σ,j we have

$$\left|\langle\,\cdot\,\rangle - \frac{\sqrt{\frac{2}{\pi}}^9}{[(1-K_t)^{18},(1+K_t)^{18}]} \int_{\mathbb{R}^9} d^9x e^{-2\sum_{Jj}x^2_{Jj}} \prod_{k=1}^N \lambda^r_{J_k\sigma_k j_k}(\{x_{Jj}+p_{Jj}/T\})\right|$$

$$= \Big| \frac{\sqrt{\frac{2}{\pi}}^9}{[(1-K_t)^{18},(1+K_t)^{18}]} \sum_{\{n_{J\sigma j}\}\neq\{0\}} e^{\frac{2}{t}\sum_{Jj}[(p^+_{Jj}-i\pi n^+_{Jj})^2+(p_{Jj}-i\pi n_{Jj})^2-(p^+)^2_{Jj}-p^2_{Jj}]} \times$$
$$\times \int_{\mathbb{R}^9} d^9x e^{-2\sum_{Jj} x^2_{Jj}} \prod_{k=1}^N \lambda^r_{J_k\sigma_k j_k}(\{x_{Jj}+(p_{Jj}-i\pi n_{Jj})/T\})|$$
$$\leq (\frac{2}{t})^N \Big| \frac{\sqrt{\frac{2}{\pi}}^9}{(1-K_t)^{18}} \sum_{\{n_{J\sigma j}\}\neq\{0\}} e^{-\frac{\pi^2}{t}\sum_{J\sigma j} n^2_{J\sigma j}} \int_{\mathbb{R}^9} d^9x e^{-2\sum_{Jj} x^2_{Jj}} \times$$
$$\times \prod_{k=1}^N [e^{\frac{r}{2}\ln(|\det(\{Tx_{Jj}+(p_{Jj}-i\pi n_{Jj})\})|)} + e^{\frac{r}{2}\ln(|\det(\{Tx_{Jj}+t\delta_{(Jj),(J_k j_k)}+(p_{Jj}-i\pi n_{Jj})\})|)}]\Big|. \tag{4.13}$$

Let w_{Jj} be a matrix of complex numbers and define the norm $\|w\|^2 \doteq \sum_{Jj}|w_{Jj}|^2$ so that in particular $||w_1+w_2|| \leq ||w_1||+||w_2||$ and $|w_{Jj}| \leq \|w\|$ for all J,j. Now $\det(\{w_{Jj}\})$ is a linear combination of six monomials of the form $w_{J_1j_1}w_{J_2j_2}w_{J_3j_3}$ so that $|\det(\{w_{Jj}\})| \leq 6\,\|w\|^3$. In particular, $|\det(\{Tx_{Jj}+(p_{Jj}-i\pi n_{Jj})\})| \leq 6(T\,\|x\|+\|p\|+\pi\,\|n\|)^3$ and $|\det(\{Tx_{Jj}+t\delta_{(Jj),(J_kj_k)}/2+(p_{Jj}-i\pi n_{Jj})\})| \leq 6(T\,\|x\|+t+\|p\|+\pi\,\|n\|)^3$. Invoking this result into (4.13) we find

$$\leq (\frac{4}{t})^N \Big| \frac{\sqrt{\frac{2}{\pi}}^9}{(1-K_t)^{18}} \sum_{\{n_{J\sigma j}\}\neq\{0\}} e^{-\frac{\pi^2}{t}\sum_{J\sigma j} n^2_{J\sigma j}} \int_{\mathbb{R}^9} d^9x e^{-2\|x\|^2} e^{\frac{Nr}{2}\ln(6[T\|x\|+t+\|p\|+\pi\|n\|]^3)}\Big|$$
$$\leq (\frac{4\,6^{r/2}}{t})^N \Big| \frac{\sqrt{\frac{2}{\pi}}^9}{(1-K_t)^{18}} \sum_{\{n_{J\sigma j}\}\neq\{0\}} e^{-\frac{\pi^2}{t}\sum_{J\sigma j} n^2_{J\sigma j}} \times$$
$$\times \int_{\mathbb{R}^9} d^9x e^{-2\|x\|^2}[\frac{1}{4}+t\,\|x\|^2+t+\|p\|+\pi\,\|n\|]^{[\frac{3Nr}{2}]+1}\Big| \tag{4.14}$$

where $[3Nr/2]$ is the Gauss bracket of a real number (largest integer smaller than or equal to $3Nr/2$) and in the last step we have used the elementary estimate $x \leq x^2+1/4$ valid for any real number x. The integral in the last line of (4.14) can be evaluated exactly by invoking the binomial theorem. Consider the integrals of the form

$$I_k \doteq \sqrt{\frac{2}{\pi}}^m \int_{\mathbb{R}^m} d^mx e^{-2\|x\|^2}\,\|x\|^{2k} \tag{4.15}$$

for any positive integer m. Switching to polar coordinates one easily proves the recursion formula

$$I_k = \frac{m+2(k-1)}{4} I_{k-1} \tag{4.16}$$

and since $I_0 = 1$ we find

$$I_k = \frac{(\frac{m}{2}+k-1)!}{2^k(\frac{m}{2})!} \text{ if } m \text{ even}$$
$$I_k = \frac{(m-1+2k)!\;(\frac{m-1}{2})!}{8^k(m-1)!(\frac{m-1}{2}+k)!} \text{ if } m \text{ odd} \tag{4.17}$$

Using the elementary estimate $e(n/e)^n \leq n! \leq e((n+1)/e)^{n+1}$ we find for $0 \leq k \leq n$ and $n \geq 2$ that

$$
\begin{aligned}
I_k &\leq e(\frac{m+2n}{2e})^{m/2}(\frac{m+2n}{4e})^k \doteq C_{m,n}(\frac{m+2n}{4e})^k \text{ if } m \text{ even} \\
I_k &\leq \frac{m-1}{2e}\frac{(\frac{m-1}{2})!}{(m-1)!}(\frac{m+2n}{m-1})^m(\frac{m+2n}{4(m-1)})^k =: C_{m,n}(\frac{m+2n}{4(m-1)})^k \text{ if } m \text{ odd}
\end{aligned} \tag{4.18}
$$

In our case $m = 9$ and $n = [\frac{3Nr}{2}] + 1$. Thus, we can finish the estimate of (4.14) with

$$
\begin{aligned}
&| < . > - \frac{\sqrt{\frac{2}{\pi}}^9}{[(1-K_t)^{18}, (1+K_t)^{18}]} \int_{\mathbb{R}^9} d^9x e^{-2\sum_{Jj} x_{Jj}^2} \prod_{k=1}^N \lambda^r_{J_k\sigma_k j_k}(\{x_{Jj} + p_{Jj}/T\})| \\
&\leq \frac{(\frac{4\, 6^{r/2}}{t})^N C_{9,[\frac{3Nr}{2}]+1}}{(1-K_t)^{18}} \sum_{\{n_{J\sigma j}\}\neq\{0\}} e^{-\frac{\pi^2}{t}\sum_{J\sigma j} n_{J\sigma j}^2} \times \\
&\times [\frac{1}{4} + t\frac{9 + 2([\frac{3Nr}{2}]+1)}{32} + t + \|p\| + \pi\,\|n\|]^{[\frac{3Nr}{2}]+1}
\end{aligned} \tag{4.19}
$$

which is obviously of order $O(t^\infty)$. We can give a bound independent of p since in our applications $\|p\|$ can be bounded by a constant of the order of t^α.

Let us summarize our findings in the form of a theorem.

Theorem 4.1. *Let $\|p(v)\|^2 \doteq \sum_{Jj} p_{Jj}(v)^2$. Suppose that there exists a positive constant K such that $\sup_{v\in V(\gamma), m\in\mathcal{M}} \|p(v)\| =: \|p\| \leq K$ is uniformly bounded. Then for small t*

$$
\langle\cdot\rangle = \frac{\sqrt{\frac{2}{\pi}}^9}{[(1-K_t)^{18}, (1+K_t)^{18}]} \int_{\mathbb{R}^9} d^9x e^{-2\sum_{Jj} x_{Jj}^2} \prod_{k=1}^N \lambda^r_{J_k\sigma_k j_k}(\{x_{Jj} + p_{Jj}/T\}) + O(t^\infty) \tag{4.20}
$$

independently of $m \in \mathcal{M}, v \in V(\gamma)$.

Expansion of the remaining integral:

It remains to compute the power expansion (in T) of the remaining integral in (4.20) and to show that at each order the remainder is smaller than the given order. We will see that only even powers of T contribute so that this expansion is actually an expansion in t. The basic reason is that the expansion of the integrand in powers of T is at the same time an expansion in powers of x_{Jj} as is obvious from the explicit form of the functions $\lambda^r(\{x_{Jj}\})$. These powers of x_{Jj} are integrated against the Gaussian $e^{-2\|x\|^2}$ which is an even function under the reflection $x_{Jj} \to -x_{Jj}$ whence the integral for odd powers (an odd function under reflection) must vanish. We will not be able to show that the integral in (4.20), which certainly converges for any p_{Jj}, t (just set $\|n\| = 0$ in above estimate), can be expanded into an infinite series in powers of t, rather our estimates will be only good enough in order to show that there is a maximal order n_0 (which becomes infinite as $t \to 0$) in the sense that the remainder at order n is smaller than the given order for all $n \leq n_0$. We will use rather coarse estimates which could possibly be much improved in order to raise the value of n_0 derived here but for all practical purposes the analysis described below will be sufficient since n_0 is anyway a rather large positive integer.

Consider once more the function $\lambda^r_{J\sigma j}(x+p/T)$: Let us introduce $q \doteq pt^{-\alpha}$ which is of order unity and $s = t^{1/2-\alpha}$. Then

$$\lambda^r_{J\sigma j}(x+p/T) = 2|\det(p)|^{r/2}\frac{|\det(1+q^{-1}xs)|^{r/2} - |\det(1+q^{-1}xs+q^{-1}\delta_{Jj}sT/2)|^{r/2}}{t} \tag{4.21}$$

Now for any matrix A we have $\det(1+A) = 1+\mathrm{Tr}(A)+\frac{1}{2}[(\mathrm{Tr}(A))^2 - \mathrm{Tr}(A^2)]+\det(A) =: 1+z'_A$ and so $\det(1+A)^2 = 1+2z'_A+(z'_A)^2 =: 1+z_A =: y_A \geq 0$. Let $y \doteq 1+z_{q^{-1}xs}$ and $y_1 \doteq 1+z_{q^{-1}[xs+\sigma\delta_{Jj}sT]}$. Then (4.21) becomes

$$\lambda^r_{J\sigma j}(x+p/T)| = \frac{2|\det(p)|^{r/2}}{t}[y^{r/4} - y_1^{r/4}] \tag{4.22}$$

and we should expand $y^{r/4}, y_1^{r/4}$ around $y = y_1 = 1$. We now invoke our knowledge that $0 < r \leq 1$ is a rational number, so we find positive integers $M > L > 0$ without common prime factor such that $r/4 = L/M$. Let us define recursively

$$\begin{aligned} f^{(0)}_{L/M}(y) &\doteq y^{L/M}, \\ f^{(n+1)}_{L/M}(y) &\doteq \frac{f^{(n)}_{L/M}(y) - f^{(n)}_{L/M}(1)}{y-1}. \end{aligned} \tag{4.23}$$

It follows from this definition that

$$f^{(0)}_{L/M}(y) = \sum_{k=0}^{n} f^{(k)}_{L/M}(1)[y-1]^k + f^{(n+1)}_{L/M}(y)[y-1]^{n+1}. \tag{4.24}$$

Lemma 4.1. *We have*

$$f^{(k)}_{L/M}(1) = (L/M, k) \tag{4.25}$$

where

$$(L/M,k) \doteq \frac{(L/M)(L/M-1)\dots(L/M-k+1)}{k!} = (-1)^{k+1}\frac{L}{M}\frac{M-L}{2M}\frac{2M-L}{3M}\dots\frac{(k-1)M-L}{kM}$$

and the following recursion holds for all $n \geq 1$

$$f^{(n+1)}_{L/M}(y) = \frac{\sum_{k=1}^{L-1} f^{(n)}_{k/M}(y) - \sum_{l=1}^{n} f^{(l)}_{L/M}(1)\sum_{k=1}^{M-1} f^{(n-l+1)}_{k/M}(y)}{\sum_{k=0}^{M-1} f^{(0)}_{k/M}(y)}. \tag{4.26}$$

The proof of the lemma consists in a straightforward Taylor expansion (first part) and an induction (second part) and will not be reproduced here.

The motivation for the derivation of this recursion is that it allows us to estimate $|f^{(n+1)}_{L/M}(y)|$ once we have an estimate for all the $|f^{(l)}_{k/M}(y)|$ with $0 \leq k \leq M-1, 0 \leq l \leq n$.

Lemma 4.2. *For all $0 < L \leq M$, $n \geq 0$ we have*

$$|f^{(n)}_{L/M}(y)| \leq (1+y)(\beta M)^n \tag{4.27}$$

where $\beta > 1$ is any positive number satisfying $\beta \geq 1+\frac{\beta}{\beta-1}$, e.g. $\beta = 3$.

This lemma can be proven by induction, using the results of the previous one.

Using the expansion (4.24) and the fact that y is a polynomial in the x_{Jj} it is possible evaluate the Gaussian integrals over the first n terms the last one of which is obviously at least of order s^n. We would like to know at which order n_0 the remaining term in (4.24) is no longer of order at least s^{n_0+1}.

To that end recall that $y = 1 + 2z + z^2$ where $z = \mathrm{Tr}(A) + \frac{1}{2}[(\mathrm{Tr}(A))^2 - \mathrm{Tr}(A^2)] + \det(A)$ and $A_{jk} = s\sum_J (q^{-1})_{Jj} x_{Jk}$. We now have the following basic estimates

$$\begin{aligned}
|\,\mathrm{Tr}(A)| &= s|\sum_{Jj} q^{-1}_{Jj} x_{Jj}| \le s||q^{-1}||\ \|x\| \\
|(q^{-1}x)_{jk}| &= |\sum_J q^{-1}_{Jj} x_{Jk}| \le \sqrt{\sum_J [q^{-1}_{Jj}]^2}\sqrt{\sum_J [x_{Jk}]^2} \\
|\,\mathrm{Tr}(A^2)| &= s^2|\sum_{jk}(q^{-1}x)_{jk}(q^{-1}x)_{kj}| \le s^2|\sum_{jk}|(q^{-1}x)_{jk}|\ |(q^{-1}x)_{kj}| \\
&\le s^2[\sum_j \sqrt{\sum_J [q^{-1}_{Jj}]^2}\sqrt{\sum_J [x_{Jj}]^2}][\sum_k \sqrt{\sum_J [q^{-1}_{Jk}]^2}\sqrt{\sum_J [x_{Jk}]^2}] \\
&\le s^2[\sqrt{\sum_j \sqrt{\sum_J [q^{-1}_{Jj}]^2}^2}\sqrt{\sum_j \sqrt{\sum_J [x_{Jj}]^2}^2}\ ]^2 \\
&\le s^2||q^{-1}||^2\ \|x\|^2 \\
|\det(A)| &\le 6s^3||q^{-1}x||^3 \le 6s^3||q^{-1}||^3\ \|x\|^3
\end{aligned}$$

where in the first line we have made use of the Cauchy-Schwarz inequality for the inner product $< x, x' >= \sum_{Jj} x_{Jj}x'_{Jj}$, in the second for the inner product $< x, x' >= \sum_J x_J x'_J$, in the fourth line for the inner product $< x, x' >= \sum_j x_j x'_j$ and finally in the last line we have used the estimate derived between equations (4.13) and (4.14). These estimates imply that

$$\begin{aligned}
|z| &\le s\left\|q^{-1}\right\|\ \|x\| + s^2\left\|q^{-1}\right\|^2\ \|x\|^2 + 6|\det(q^{-1})|\ \|x\|^3 =: u(\|x\|), \\
|y-1| &\le 2u + u^2 =: P(\|x\|)
\end{aligned}$$

and $P(\|x\|)$ is a polynomial of sixth order in $\|x\|$.

We are now ready to estimate the Gaussian integral over the remainder:

$$\begin{aligned}
E_n &\doteq \left|\sqrt{\frac{2}{\pi}}^9 \int_{\mathbb{R}^9} d^9x e^{-2\|x\|^2} f^{(n+1)}_{L/M}(y)[y-1]^{n+1}\right| \\
&\le \sqrt{\frac{2}{\pi}}^9 (3M)^{n+1}\int_{\mathbb{R}^9} d^9x e^{-2\|x\|^2}[(P(\|x\|))^{n+2} + 2(P(\|x\|))^{n+1}]. \qquad (4.28)
\end{aligned}$$

Consider an arbitrary polynomial in $\|x\|$ of the form

$$P(x) = \sum_{k=0}^{l} a_k \|x\|^k .$$

By the multinomial theorem

$$(P(x))^n = \sum_{n_0+..+n_l=n} \frac{n!}{(n_0!)..(n_l)!}[\prod_{k=0}^{l} a_k^{n_k}] \; \|x\|^{\sum_{k=0}^{l} kn_k} .$$

Let us consider Gaussian integrals of the form

$$\sqrt{\frac{2}{\pi}}^m \int_{\mathbb{R}^m} d^m x e^{-2\|x\|^2} \; \|x\|^n = V_{m-1}\sqrt{\frac{2}{\pi}}^m \int_0^\infty dr e^{-2r^2} r^{n+m-1} =: V_{m-1}\sqrt{\frac{2}{\pi}}^m J_{n+m-1}$$

where $V_m = 2\pi^{m/2}/\Gamma(m/2)$ is the volume of S^m. Now

$$\begin{aligned} J_n &= \frac{\sqrt{2\pi}}{4} 2^{-3n/2} \frac{n!}{\frac{n}{2}!} \text{ for } n \text{ even}, \\ J_n &= \frac{1}{4} 2^{-(n-1)/2} (\frac{n-1}{2}!) \text{ for } n \text{ odd}, \end{aligned} \tag{4.29}$$

and one immediately checks that

$$J_n \leq \frac{\sqrt{2\pi}}{4} \frac{[\frac{n}{2}]!}{2^{[\frac{n}{2}]}}$$

where [.] again denotes the Gauss bracket. Using the above used estimate for the factorial $n! \leq e(\frac{(n+1)}{e})^{n+1}$ we may further estimate

$$J_n \leq \frac{e\sqrt{2\pi}}{4} \frac{(\frac{n+1}{2e})^{\frac{n+1}{2}}}{2^{\frac{n-1}{2}}} = \frac{e\sqrt{2\pi}}{4} 2^{-n} (\frac{n+1}{e})^{\frac{n+1}{2}}$$

where we used $\frac{n-1}{2} \leq [\frac{n}{2}] \leq \frac{n}{2}$. Finally, if $n \leq n_M$ then

$$J_n \leq \frac{e\sqrt{2\pi}}{4} 2^{-n} (\frac{n_M+1}{e})^{\frac{n+1}{2}}. \tag{4.30}$$

Combining these results we obtain the final estimate

$$\begin{aligned} &\sqrt{\frac{2}{\pi}}^m \int_{\mathbb{R}^m} d^m x e^{-2\|x\|^2} P(x)^n = V_{m-1}\sqrt{\frac{2}{\pi}}^m \frac{e\sqrt{2\pi}}{4} \sum_{n_0+..+n_l=n} \frac{n!}{(n_0!)..(n_l)!}[\prod_{k=0}^{l} a_k^{n_k}] J_{\sum_{k=0}^{l} kn_k+m-1} \\ &\leq V_{m-1}\sqrt{\frac{2}{\pi}}^m \frac{e\sqrt{2\pi}}{4} \sum_{n_0+..+n_l=n} \frac{n!}{(n_0!)..(n_l)!}[\prod_{k=0}^{l} a_k^{n_k}] 2^{-(\sum_{k=0}^{l} kn_k+m-1)} (\frac{m+ln}{e})^{\frac{\sum_{k=0}^{l} kn_k+m-1+1}{2}} \\ &= V_{m-1}\sqrt{\frac{2}{\pi}}^m \frac{e\sqrt{2\pi}}{2} (\frac{m+ln}{4e})^{\frac{m}{2}} \sum_{n_0+..+n_l=n} \frac{n!}{(n_0!)..(n_l)!}[\prod_{k=0}^{l} (a_k \sqrt{\frac{m+ln}{4e}}^k)^{n_k}] \\ &= V_{m-1}\sqrt{\frac{2}{\pi}}^m \frac{e\sqrt{2\pi}}{2} (\frac{m+ln}{4e})^{\frac{m}{2}} [\sum_{k=0}^{l} a_k \sqrt{\frac{1+ln}{4e}}^k \;]^n \end{aligned}$$

$$=: K_{m,l}(\frac{m+ln}{4e})^{\frac{m}{2}} P(\sqrt{\frac{m+ln}{4e}}) \tag{4.31}$$

since $\sum_{k=0}^{l} kn_k \le ln = n_M - m$ for any configuration of the n_k subject to the constraint $n_0+..+n_l = n$.

In our case we have $m = 9, l = 6$ and thus we can bound the remainder (4.28) from above:

$$E_n \le K_{9,6}(3M)^{n+1}\Big[(\frac{9+6(n+2)}{4e})^{\frac{9}{2}}(P(\sqrt{\frac{9+6(n+2)}{4e}})^{n+2} + 2(\frac{9+6(n+1)}{4e})^{\frac{9}{2}}(P(\sqrt{\frac{1+6(n+1)}{4e}})^{n+1}\Big]. \tag{4.32}$$

For small n the error E_n is the number s^{n+1} times a constant of order unity. For large n, however, the error becomes comparable to the order of accuracy (in powers of s) that we are interested in itself. The value $n = n_0$ from where onwards it does not make sense any longer to compute corrections can be estimated from the condition

$$E_{n+1}/E_n \ge 1. \tag{4.33}$$

Due to the complicated structure of (4.32) the precise value of n_0 cannot be computed analytically but its order of magnitude can be obtained under the self-consistency assumption that n_0 is quite large so that the change of $P(\sqrt{(9+6(n_0+2))/(4e)})$ as we change n_0 by 1 is much smaller than its value. A tedious but straightforward estimate shows that under this assumption

$$n_0 = \frac{4e(\frac{\tau_0(M)}{s||q^{-1}||})^2 - 9}{6} - 3 \tag{4.34}$$

where $\tau_0(M)$ is of order unity. Thus n_0 is a very large number if $||q^{-1}||$ is of order unity and s is tiny. Moreover,

$$\delta P = 2(u+1)(1+2\tau+18\tau^2)\delta\tau = 6(u+1)u\delta\tau/\tau \le 6P\frac{\delta\tau}{\tau}. \tag{4.35}$$

But under the change $\delta n = 1$

$$\delta\tau \approx \frac{d\tau}{dn}\delta n = \frac{\tau}{9(9+2n)} \tag{4.36}$$

whence

$$(\frac{\delta P}{P})_{n=n_0} \le \frac{2}{3(9+2n_0)} \ll 1 \tag{4.37}$$

as desired since n_0 is a large number.

Let us now finally go back to our desired expectation value (4.20) which we would like to compute up to some order $n < n_0$ in s. Let again $y \doteq 1 + z_{q^{-1}xs} = 1 + z$ and $y_{J\sigma j} \doteq 1 + z_{q^{-1}[xs+\delta_{Jj}sT/2]} = 1 + z_{J\sigma j}$ with $z_A = (z'_A)^2 + 2z'_A,\ z'_A = \mathrm{Tr}(A) + \frac{1}{2}[(\mathrm{Tr}(A))^2 - \mathrm{Tr}(A^2)] + \det(A)$ for any matrix A and recall our convention $r/4 = L/M$. Thus (4.22) becomes up to order n

$$\lambda^r_{J\sigma j}(x+p/T) = \frac{2|\det(p)|^{2L/M}}{t}[y^{L/M} - y^{L/M}_{J\sigma j}] \tag{4.38}$$

$$= \frac{2|\det(q)|^{2L/M} t^{6L/M\alpha}}{t} \{[(y - y_{J\sigma j}) \sum_{k=1}^{n} f_{L/M}^{(k)}(1) \sum_{l=0}^{k-1} (y-1)^l (y_{J\sigma j} - 1)^{k-1-l}]$$
$$+ [f_{L/M}^{(n+1)}(y)(y-1)^{n+1} - f_{L/M}^{(n+1)}(y_{J\sigma j})(y_{J\sigma j} - 1)^{n+1}]\}$$

In order to compute (4.20) up to order n with respect to s we have to plug the expansions (4.38) into formula (4.20) and to collect all the contributions up to order s^n. The integral over the remainder is then still smaller as long as $n < n_0$ as shown above. In the present work we are interested only in the leading order (classical limit) and next to leading order (first quantum correction) as well as in an estimate of the error at the next to leading order.

A laborious but straightforward power counting reveals that

$$\lambda^r_{J\sigma j} = \frac{sT}{t}(1 + sx + (sx)^2 + O(sT)) \tag{4.39}$$

where the notation just means that $\lambda^r_{J\sigma j}$ is a polynomial in x_{Jj} of order two where the monoms of order $0, 1, 2$ come with a power of s of the order indicated *or higher*. We thus see that if we wish to keep only terms up to order $(sT/t)^N$ and $(sT/t)^N s^2$ in $\prod_{k=1}^N \lambda^r_{J_k\sigma_k j_k}(x + p/T)$ it will be sufficient to do the following: For the term of order $(sT/t)^N$ keep only the terms proportional to x^0 in each of the factors of the form (4.39) and for term of order $(sT/t)^N s^2$ keep either 1. only the terms proportional to x^2 in one of the factors of the form (4.39) and only the terms of order x^0 in the others or 2. only the terms proportional to x^1 in two of the factors of the form (4.39) and only the terms of order x^0 in the others. Clearly terms of order $(sT/t)^N s$ do not survive since they are linear in x and integrate to zero against the Gaussian.

In estimating the error that we make notice that there are two errors, one coming from neglecting all higher orders in (4.39) and one from the remainder in the expansion (4.38). As for the first error, notice that all Gaussian integrals are of order unity so that the indicated powers of t correctly display the error (compared to $(sT/t)^N s^2$) as of higher order in s. As for the second error we can use the expansion (4.38) up to some order $n' > 2$ until $s^{n'+1} \ll sTs^2$ in view of the estimate (4.32). The minimal value of n' depends on the value of α. For instance, if $\alpha = 1/6$ as indicated by [**?**] then $s = t^{1/3}$ so that $s^{n'-2} = t^{(n'-2)/3} \ll T = t^{1/2}$ means $n' > 2 + 3/2$ so the minimal value would be $n' = 4$ in this case. This value is well below $n_0 \gg 1$ so that the error is indeed of higher order in s as compared to $(sT/t)^N s^2$.

With these things said we can now actually compute the first contributing correction to the classical limit. We will not bother with the higher order corrections since we just showed that they can be bounded by terms of sub-leading order as compared to $(sT/t)^N s^2$. In particular, we will replace the $O(t^\infty)$ corrections by zero in (4.20). We then have

$$\begin{aligned}\langle \cdot \rangle = \sqrt{\frac{2}{\pi}}^9 \int_{\mathbb{R}^9} d^9x e^{-2\|x\|^2} \Big\{ & [\prod_{k=1}^N \lambda^r_{J_k\sigma_k j_k}(x + p/T)_{|x^0}] \\ & + [\sum_{l=1}^N \lambda^r_{J_l\sigma_l j_l}(x + p/T)_{|x^2}] \prod_{k\neq l} \lambda^r_{J_k\sigma_k j_k}(x + p/T)_{|x^0}] \\ & + [\sum_{1\le l<m\le N} \lambda^r_{J_l\sigma_l j_l}(x + p/T)_{|x^1}] \lambda^r_{J_m\sigma_m j_m}(x + p/T)_{|x^1}] \prod_{k\neq l,m} \lambda^r_{J_k\sigma_k j_k}(x + p/T)_{|x^0}] \Big\}\end{aligned} \tag{4.40}$$

$$+ O(t^{(N[3r/2-1]\alpha} sT) \tag{4.41}$$

where the restrictions mean the ones to the appropriate powers of x as derived above. It remains to explicitly compute these restrictions and to do the Gaussian integrals. According to what we have said above we write

$$\begin{aligned}
\lambda^r_{J\sigma j}(x+p/T) &= O(t^{[3r/2-1]\alpha} sT) + 2|\det(q)|^{r/2} t^{[3r/2-1]\alpha}\{[f^{(1)}_{r/4}(1)(\frac{y-y_{J\sigma j}}{sT})_{|x^0}] \\
&+ [f^{(1)}_{r/4}(1)(\frac{y-y_{J\sigma j}}{sT})_{|x^1}] + f^{(2)}_{r/4}(1)(\frac{y-y_{J\sigma j}}{sT})_{|x^0}((y-1)_{|x^1} + (y_{J\sigma j}-1)_{|x^1})] \\
&+ [f^{(1)}_{r/4}(1)(\frac{y-y_{J\sigma j}}{sT})_{|x^2}] + f^{(2)}_{r/4}(1)(\frac{y-y_{J\sigma j}}{sT})_{|x^0}((y-1)_{|x^2} + (y_{J\sigma j}-1)_{|x^2}) \\
&+ f^{(3)}_{r/4}(1)(\frac{y-y_{J\sigma j}}{sT})_{|x^0}(((y-1)_{|x^1})^2 + ((y_{J\sigma j}-1)_{|x^1})^2 + (y-1)_{|x^1}(y_{J\sigma j}-1)_{|x^1})]\}
\end{aligned} \tag{4.42}$$

And furthermore

$$\begin{aligned}
y-1 &= 2sq^{-1}_{Mm}x_{Mm} + s^2(2q^{-1}_{Mm}q^{-1}_{Nn} - q^{-1}_{Mn}q^{-1}_{Nm})x_{Mm}x_{Nn} + O(s^3) \\
&=: sC^{Mm}x_{Mm} + s^2 C^{Mm,Nn}x_{Mm}x_{Nn} + O(s^3) \\
y_{J\sigma j}-1 &= 2s\,\mathrm{Tr}(q^{-1}x) + s^2[2\,\mathrm{Tr}(q^{-1}x)^2 - \mathrm{Tr}(q^{-1}xq^{-1}x)] + O(sT) \\
&=: sC^{Mm}x_{Mm} + s^2 C^{Mm,Nn}x_{Mm}x_{Nn} + O(sT) \\
\frac{y_{J\sigma j}-y}{sT} &= q^{-1}_{Jj} + s(2q^{-1}_{Jj}q^{-1}_{Mm} - q^{-1}_{Jm}q^{-1}_{Mj})x_{Mm} + \frac{s^2}{2}[\det(q^{-1})\epsilon_{jmn}\epsilon_{JMN} + q^{-1}_{Jj}(q^{-1}_{Mm}q^{-1}_{Nn} - q^{-1}_{Mn}q^{-1}_{Nm}) \\
&\qquad + 2q^{-1}_{Mm}(q^{-1}_{Jj}q^{-1}_{Nn} - q^{-1}_{Jn}q^{-1}_{Nj})]x_{Mm}x_{Nn} \\
&=: C_{J\sigma j} + sC^{Mm}_{J\sigma j}x_{Mm} + s^2 C^{Mm,Nn}_{J\sigma j}x_{Mm}x_{Nn}.
\end{aligned} \tag{4.43}$$

We can therefore simplify (4.42) to

$$\begin{aligned}
\lambda^r_{J\sigma j}(x+p/T) &= O(t^{[3r/2-1]\alpha} sT) + 2|\det(q)|^{r/2} t^{[3r/2-1]\alpha}\{[f^{(1)}_{r/4}(1)C_{J\sigma j}] \\
&+ s[f^{(1)}_{r/4}(1)C^{Mm}_{J\sigma j} + 2f^{(2)}_{r/4}(1)C_{J\sigma j}C^{Mm}]x_{Mm} \\
&+ s^2[f^{(1)}_{r/4}(1)C^{Mm,Nn}_{J\sigma j} + 2f^{(2)}_{r/4}(1)C_{J\sigma j}C^{Mm,Nn} + 3f^{(3)}_{r/4}(1)C_{J\sigma j}C^{Mm}C^{Nn}]x_{Mm}x_{Nn}\} \\
&=: O(t^{[3r/2-1]\alpha} sT) + 2|\det(q)|^{r/2} t^{[3r/2-1]\alpha}\{D_{J\sigma j}(r) + sD^{Mm}_{J\sigma j}(r)x_{Mm} \\
&+ s^2 D^{Mm,Nn}_{J\sigma j}(r)x_{Mm}x_{Nn}\}.
\end{aligned} \tag{4.44}$$

Putting everything together now yields the following theorem.

Theorem 4.2. *For the classical limit and lowest order quantum corrections of expectation values of monomials of the operators $\widehat{q}_{J\sigma j}(r)$ for topologically cubic graphs we have*

$$\begin{aligned}
\frac{\langle \psi^t_{\{g_{J\sigma j}\}}, \prod_{k=1}^N \widehat{q}_{J_k\sigma_k j_k}(r_k)\psi^t_{\{g_{J\sigma j}\}}\rangle}{||\psi^t_{\{g_{J\sigma j}\}}||^2} &= (2|\det(q)|^{r/2} t^{[3r/2-1]\alpha})^N \times \\
&\times \{[\prod_{k=1}^N D_{J_k\sigma_k j_k}(r)] + \frac{s^2}{4}\sum_{M,m}[\sum_{l=1}^N D^{Mm,Mm}_{J_l\sigma_l j_l}(r)\prod_{k\neq l} D_{J_k\sigma_k j_k}(r))
\end{aligned}$$

$$+\sum_{1\le i<l\le N} D^{Mm}_{J_i\sigma_i j_i}(r) D^{Mm}_{J_l\sigma_l j_l}(r) \prod_{k\neq l,i} D_{J_k\sigma_k j_k}(r)]\} \tag{4.45}$$

where the constants are given by

$$\begin{aligned}
C^{Mm} &= 2q^{-1}_{Mm},\\
C^{Mm,Nn} &= 2q^{-1}_{Mm}q^{-1}_{Nn} - q^{-1}_{Mn}q^{-1}_{Nm},\\
C_{J\sigma j} &= q^{-1}_{Jj},\\
C^{Mm}_{J\sigma j} &= (2q^{-1}_{Jj}q^{-1}_{Mm} - q^{-1}_{Jm}q^{-1}_{Mj}),\\
C^{Mm,Nn}_{J\sigma j} &= \frac{1}{2}[\det(q^{-1})\epsilon_{jmn}\epsilon_{JMN} + q^{-1}_{Jj}(q^{-1}_{Mm}q^{-1}_{Nn} - q^{-1}_{Mn}q^{-1}_{Nm}) + 2q^{-1}_{Mm}(q^{-1}_{Jj}q^{-1}_{Nn} - q^{-1}_{Jn}q^{-1}_{Nj})],\\
D_{J\sigma j}(r) &= f^{(1)}_{r/4}(1)C_{J\sigma j},\\
D^{Mm}_{J\sigma j}(r) &= f^{(1)}_{r/4}(1)C^{Mm}_{J\sigma j} + 2f^{(2)}_{r/4}(1)C_{J\sigma j}C^{Mm},\\
D^{Mm,Nn}_{J\sigma j}(r) &= f^{(1)}_{r/4}(1)C^{Mm,Nn}_{J\sigma j} + 2f^{(2)}_{r/4}(1)C_{J\sigma j}C^{Mm,Nn} + 3f^{(3)}_{r/4}(1)C_{J\sigma j}C^{Mm}C^{Nn},
\end{aligned}$$

and the $f^{(k)}_{r/4}(1) = (r/4, k)$ *are simply the binomial coefficients.*
The first correction is small as long as $\alpha < 1/2$. *The error as compared to the first quantum correction of order* $O(t^{(N[3r/2-1]\alpha}s^2)$ *is a constant of order unity times* $t^{(N[3r/2-1]\alpha}sT$ *and thus small as long as* $0 < \alpha$.

So far we did not look at the classical limit and the first quantum corrections of (powers of) the volume operator itself but it is clear that it can be analyzed by similar methods, in fact, the analysis is even much simpler because we just need to expand $\lambda^r(x + p/T)$ in powers of s without dividing by t and thus will have to do an expansion in terms of $y - 1$ of one order less than for $\lambda^r_{J\sigma j}(x+p/T)$. Clearly the classical order will be that of $|\det(p)|^{r/2} = |\det(q)|^{r/2}t^{3r\alpha/2} = O(t^{3r\alpha/2})$ and the first quantum correction will be of order $O(t^{3r\alpha/2}s^2)$. We thus have, in expanding up to second order in $y - 1$, where $y = \det(1 + sq^{-1}x)^2$ as before

$$\lambda^r(x + p/T) = |\det(q)|^{r/2}t^{3r\alpha/2}\Big\{1 + sf^{(1)}_{r/4}(1)C^{Mm}x_{Mm} \tag{4.46}$$

$$+ s^2[f^{(2)}_{r/4}(1)C^{Mm,Nn} + f^{(1)}_{r/4}(1)C^{Mm}C^{Nn}]x_{Mm}x_{Nn}\Big\} + O(t^{3r\alpha/2}s^3). \tag{4.47}$$

Thus we obtain an analogue of theorem 4.2 above:

Theorem 4.3. *For the classical limit and lowest order quantum corrections of expectation values of powers of the volume operators* $\widehat{\sqcup}^r_v$ *for topologically cubic graphs we have*

$$\frac{\langle\psi^t_{\{g_{J\sigma j}\}}, \widehat{\sqcup}^r_v\psi^t_{\{g_{J\sigma j}\}}\rangle}{||\psi^t_{\{g_{J\sigma j}\}}||^2} = |det(q)|^{r/2}t^{3r\alpha/2}\{1 + \frac{s^2}{4}\sum_{M,m}[f^{(2)}_{r/4}(1)C^{Mm,Nn} + f^{(1)}_{r/4}(1)C^{Mm}C^{Nn}\}. \tag{4.48}$$

The first correction is small as long as $\alpha < 1/2$. *The error as compared to the first quantum correction of order* $O(t^{(N[3r/2-1]\alpha}s^2)$ *is a constant of order unity times* $t^{(N[3r/2-1]\alpha}s^3$ *and thus small as long as* $0 < \alpha$.

4.3 Application to the Hamiltonians

So far our considerations were completely general and model independent and we see that our coherent states indeed predict small quantum predictions as long as $0 < \alpha < 1/2$ and $\ell_P/L \ll 1$ with controllable error. However, now we will specialize to the case of scalar, electromagnetic and fermionic fields coupled to gravity and compute the expectation values of the relevant gravitational operators.

We recall from our companion paper [?] that on a cubic graph, the effective Hamiltonians for the scalar and the electromagnetic field are

$$H_{\rm KG}^{\rm eff} = \frac{1}{2Q_{\rm KG}} \sum_v \left(\langle \widehat{F}_{\rm kin}(v) \rangle \widehat{\pi}_v^2 - \sum_{I\sigma I'\sigma'} \langle \widehat{F}_{\rm der}^{I\sigma I'\sigma'}(v) \rangle \left(\partial^+_{e_I^\sigma} \ln U_v \right) \left(\partial^+_{e_{I'}^{\sigma'}} \ln U_v \right) - K^2 \langle \widehat{V}_v \rangle (\ln U_v)^2 \right), \tag{4.49}$$

$$H_{\rm EM}^{\rm eff} = \frac{1}{2Q_{\rm EM}} \sum_v \sum_{I\sigma I'\sigma'} \Big(\langle \widehat{F}_{\rm el}^{I\sigma I'\sigma'}(v) \rangle Y_{I\sigma} Y_{I'\sigma'} - \langle \widehat{F}_{\rm mag}^{I\sigma I'\sigma'}(v) \rangle [\sum_{\sigma_1,\sigma_2} \ln(H_{\beta^I_{\sigma;\sigma_1,\sigma_2}(v)})] \, [\sum_{\sigma_1',\sigma_2'} \ln(H_{\beta^J_{\sigma';\sigma'_1\sigma'_2}(v)})] \Big), \tag{4.50}$$

where $\langle \cdot \rangle$ denotes the expectation value in a semiclassical state for the gravitational sector, and the geometric operators are given by

$$\widehat{F}_{\rm kin}(v) = \frac{1}{\ell_P^{12}} \left[\frac{1}{8} \sum_{\sigma_1,\sigma_2,\sigma_3} \frac{\sigma_1\sigma_2\sigma_3}{3!} \epsilon_{ijk} \epsilon^{IJK} \hat{Q}^i_{I,\sigma_1}(v,\frac{1}{2}) \hat{Q}^j_{J,\sigma_2}(v,\frac{1}{2}) \hat{Q}^k_{K,\sigma_3}(v,\frac{1}{2}) \right]^\dagger \left[\ldots \right],$$

$$\widehat{F}_{\rm der}^{I\sigma I'\sigma'} = \frac{1}{4} \frac{1}{\ell_P^8} \sum_j \left[\frac{\epsilon^{IJK}}{8} \epsilon_{jkl} \sum_{\sigma_2,\sigma_3} \hat{Q}^k_{J\sigma_2}(v,\frac{3}{4}) \hat{Q}^l_{K\sigma_3}(v,\frac{3}{4}) \right] \left[\ldots {}_j \ldots \right],$$

$$\widehat{F}_{\rm el}^{I\sigma I'\sigma'}(v) = \frac{1}{4} \frac{1}{\ell_P^4} \sum_j \hat{Q}^j_{I\sigma}(v,\frac{1}{2}) \hat{Q}^j_{I'\sigma'}(v,\frac{1}{2}),$$

$$\widehat{F}_{\rm mag}^{I\sigma I'\sigma'}(v) = \frac{1}{64} \frac{1}{\ell_P^4} \sum_j \hat{Q}^j_{I\sigma}(v,\frac{1}{2}) \hat{Q}^j_{I'\sigma'}(v,\frac{1}{2}).$$

The matter fields are represented as

$$\widehat{\phi}_v = \frac{\ln U_v}{i}, \qquad \widehat{\pi}_v = i\hbar Q_{KG} Y_v, \qquad \widehat{E}_{I\sigma}(v) = i\hbar Q_{\rm EM} Y_{I\sigma}, \qquad \widehat{B}_{I\sigma;\sigma_1,\sigma_2}(v) = \frac{\ln(H_{\beta^I_{\sigma;\sigma_1,\sigma_2}(v)})}{i},$$

where the Y are invariant derivatives on U(1), U_v is a U(1) point-holonomy and H_β a U(1) holonomy around a *minimal loop*.

The Hamiltonian for a fermionic field is given by

$$\hat{H}_{D,\gamma} = -\frac{m_P}{2\ell_P^3} \sum_{v,v' \in V(\gamma)} [\hat{\theta}_B(v') \hat{\theta}_A^\dagger(v) - \hat{\theta}'_B(v') \hat{\theta}'^\dagger_A(v)] \times$$
$$\times \quad \{ \frac{1}{8} \epsilon_{ijk} \epsilon^{IJK} \sum_{\sigma_1,\sigma_2,\sigma_3} \hat{Q}^i_{I\sigma_1}(v,\frac{1}{2}) \hat{Q}^j_{J\sigma_2}(v,\frac{1}{2}) [\tau^k (h_K^{\sigma_3}(v) \delta_{v',f(e_K^{\sigma_3}(v))} - \delta_{v',v})]_{AB} \}$$

$$-\{\frac{1}{8}\epsilon_{ijk}\epsilon^{IJK}\sum_{\sigma'_1,\sigma'_2,\sigma'_3}[([h_K^{\sigma'_3}(v')]^{-1}\delta_{v,f(e_K^{\sigma'_3}(v'))}-\delta_{v,v'})\tau^k]_{AB}\hat{Q}^i_{I\sigma'_1}(v',\frac{1}{2})\hat{Q}^j_{J\sigma'_2}(v',\frac{1}{2})\}\}$$

$$-i\hbar K_0\sum_{v,v'\in V(\gamma)}\delta_{AB}\delta_{v,v'}[\hat{\theta}'_B(v')\hat{\theta}^\dagger_A(v)-\hat{\theta}_B(v')\hat{\theta}'^\dagger_A(v)]. \tag{4.51}$$

We strongly recommend to take a look at [?] where the above Hamiltonians are derived and all the ingredients are defined and discussed in detail!

We now proceed to compute the expectation values of the geometric operators in a coherent state. To this end, we will use the formulae given in theorems 4.2 and 4.3 with the appropriate values of r, N, J_k, σ_k, j_k inserted, and and perform the additional computations necessary.

4.3.1 The Kinetic Term

For $F_{\rm kin}$ we have to use theorem 4.2 with $N=6$. Employing the relation (4.2) between $\widehat{q}$ and $\widehat{Q}$ we find

$$\begin{aligned}\langle\widehat{F}_{\rm kin}\rangle = {} & \frac{1}{\ell_P^{12}}\frac{1}{(3!)^2}\left(\frac{2ta^{3r}}{r}\right)^6(2|\det(q)|^{1/4}t^{[3/4-1]\alpha})^6\ \epsilon^{J_1J_2J_3}\epsilon_{j_1j_2j_3}\epsilon^{J_4J_5J_6}\epsilon_{j_4j_5j_6}\times \\ & \times\{[\prod_{k=1}^6 D_{J_k\sigma_kj_k}(1/2)]+\frac{s^2}{4}\sum_{M,m}[\sum_{l=1}^6 D^{Mm,Mm}_{J_l\sigma_lj_l}(1/2)\prod_{k\neq l}D_{J_k\sigma_kj_k}(1/2)) \\ & +\sum_{1\le i<l\le 6}D^{Mm}_{J_i\sigma_ij_i}(1/2)D^{Mm}_{J_l\sigma_lj_l}(1/2)\prod_{k\neq l,i}D_{J_k\sigma_kj_k}(1/2)]\}.\end{aligned} \tag{4.52}$$

For $r=1/2$ we have

$$a_1\doteq f^{(1)}_{1/8}(1)=\frac{1}{8},\ a_2\doteq f^{(2)}_{1/8}(1)=-\frac{1}{8}\frac{7}{16}=-\frac{7}{128},\ a_3\doteq f^{(3)}_{1/8}(1)=\frac{7}{128}\frac{15}{24}=\frac{35}{1024}, \tag{4.53}$$

and consequently

$$\sum_{M,m}D^{Mm,Mm}_{J\sigma j}(1/2)=[a_1+3a_3]q^{-1}_{Jj}\,{\rm Tr}(q^{-2})-\frac{a_1}{2}q^{-3}_{Jj} \tag{4.54}$$

$$\begin{aligned}\sum_{Mm}D^{Mm}_{J_1\sigma_1j_1}(1/2)D^{Mm}_{J_2\sigma_2j_2}(1/2) = {} & 4[a_1+a_2]^2q^{-1}_{J_1j_1}q^{-1}_{J_2j_2}\,{\rm Tr}(q^{-2}) \\ & -2a_1[a_1+a_2](q^{-1}_{J_1j_1}q^{-3}_{J_2j_2}+q^{-1}_{J_2j_2}q^{-3}_{J_1j_1})+a_1^2q^{-2}_{J_1J_2}q^{-2}_{j_1j_2}.\end{aligned} \tag{4.55}$$

Now we have to deal with the contractions in (4.52). It is easy to see that

$$\epsilon^{J_1J_2J_3}\epsilon_{j_1j_2j_3}\epsilon^{J_4J_5J_6}\epsilon_{j_4j_5j_6}\prod_{k=1}^6 q^{-1}_{J_kj_k}]=\frac{36}{\det(q)^2},$$

$$\epsilon^{J_1J_2J_3}\epsilon_{j_1j_2j_3}\epsilon^{J_4J_5J_6}\epsilon_{j_4j_5j_6}q^{-3}_{J_lj_l})\prod_{k\neq l}q^{-1}_{J_kj_k}=\frac{12\,{\rm Tr}(q^{-2})}{\det(q)^2},$$

$$\epsilon^{J_1J_2J_3}\epsilon_{j_1j_2j_3}\epsilon^{J_4J_5J_6}\epsilon_{j_4j_5j_6}q^{-2}_{J_iJ_l}q^{-2}_{j_ij_l})\prod_{k\neq l,i}q^{-1}_{J_kj_k}=0 \text{ if } l,i\in\{1,2,3\} \text{ or } l,i\in\{4,5,6\},$$

$$\epsilon^{J_1J_2J_3}\epsilon_{j_1j_2j_3}\epsilon^{J_4J_5J_6}\epsilon_{j_4j_5j_6}q^{-2}_{J_iJ_l}q^{-2}_{j_ij_l})\prod_{k\neq l,i}q^{-1}_{J_kj_k} = \frac{4\,\mathrm{Tr}(q^{-2})}{\det(q)^2} \text{ otherwise.} \tag{4.56}$$

Using the above together with (4.54) in (4.52) yields

$$\begin{aligned}\langle\widehat{F}_{\mathrm{kin}}(v)\rangle &= \frac{a^9}{\ell_P^{12}}\frac{4^6}{(3!)^2}t^6\frac{(2|\det(q)|^{1/4}t^{[3/4-1]\alpha})^6}{\det(q)^2}\ \{36[a_1^6]+\frac{s^2}{4}\mathrm{Tr}(q^{-2})[6a_1^5(36[a_1+3a_3]-12\frac{a_1}{2})\\ &\quad+a_1^4(15(4[a_1+a_2]^2(36)-2a_1[a_1+a_2](12+12))+9a_1^2]\}\\ &= \frac{a^9t^6}{\ell_P^{12}}\frac{1}{\sqrt{\det p}}\{1+\frac{t}{4}\mathrm{Tr}(p^{-2})[(5+24a_3)+15(4[1+8a_2]^2-\frac{4}{3}[1+8a_2])+\frac{1}{4}]\}\\ &= \frac{a^9t^6}{\ell_P^{12}}\frac{1}{\sqrt{\det p}}\{1+t\frac{1707}{512}\mathrm{Tr}(p^{-2})\}.\end{aligned} \tag{4.57}$$

Let us finally transform back to the dimensionfull quantity $P=a^2p$. We get

$$\langle\widehat{F}_{\mathrm{kin}}\rangle(v) = \frac{1}{\sqrt{\det P(v)}}\left[1+\frac{\ell_P^4}{t}\frac{1707}{512}\mathrm{Tr}\,P^{-2}(v)\right].$$

4.3.2 The Derivative Term

The derivative term F_{der} requires $N=4$. From theorem 4.2 we find

$$\langle F^{J\sigma J'\sigma'}_{\mathrm{der}}\rangle = \frac{\sigma\sigma'}{4}\frac{1}{4}\left(\frac{8}{3}\right)^4\frac{t^4a^9}{\ell_P^8}(2|\det(q)|^{3/8}t^{[9/8-1]\alpha})^4\ \sum_j\epsilon^{JJ_1J_1}\epsilon_{jj_1j_2}\epsilon^{J'J_3J_4}\epsilon_{jj_3j_4}\frac{1}{16}\sum_{\sigma_1\ldots\sigma_4} \tag{4.58}$$

$$\begin{aligned}&\times\{[\prod_{k=1}^4 D_{J_k\sigma_kj_k}(3/4)]+\frac{s^2}{4}\sum_{M,m}[\sum_{l=1}^4 D^{Mm,Mm}_{J_l\sigma_lj_l}(3/4)\prod_{k\neq l}D_{J_k\sigma_kj_k}(3/4))\\ &+\sum_{1\le i<l\le 4}D^{Mm}_{J_i\sigma_ij_i}(3/4)D^{Mm}_{J_l\sigma_lj_l}(3/4)\prod_{k\neq l,i}D_{J_k\sigma_kj_k}(3/4)]\}\end{aligned} \tag{4.59}$$

For $r=3/4$ we have

$$a_1\doteq f^{(1)}_{3/16}(1)=\frac{3}{16},\ a_2\doteq f^{(2)}_{3/16}(1)=-\frac{3}{16}\frac{29}{32}=-\frac{3\cdot 29}{2^9},\ a_3\doteq f^{(3)}_{3/16}(1)=\frac{3\cdot 29}{2^9}\frac{45}{48}=\frac{3^2\cdot 5\cdot 29}{2^{13}}. \tag{4.60}$$

Furthermore the reader may verify that

$$\begin{aligned}&\sum_j\epsilon^{JJ_1J_2}\epsilon_{jj_1j_2}\epsilon^{J'J_3J_4}\epsilon_{jj_3j_4}\prod_{k=1}^4 q^{-1}_{J_kj_k}] = \frac{4q^2_{JJ'}}{\det(q)^2},\\ &\sum_j\epsilon^{JJ_1J_2}\epsilon_{jj_1j_2}\epsilon^{J'J_3J_4}\epsilon_{jj_3j_4}q^{-3}_{J_lj_l})\prod_{k\neq l}q^{-1}_{J_kj_k} = \frac{2[q^2_{JJ'}\mathrm{Tr}(q^{-2})-\delta_{JJ'}]}{\det(q)^2},\\ &\sum_j\epsilon^{JJ_1J_2}\epsilon_{jj_1j_2}\epsilon^{J'J_3J_4}\epsilon_{jj_3j_4}q^{-2}_{J_iJ_l}q^{-2}_{j_ij_l}\prod_{k\neq l,i}q^{-1}_{J_kj_k} = 0 \text{ if } l,i\in\{1,2\} \text{ or } l,i\in\{3,4\},\end{aligned}$$

$$\sum_j \epsilon^{JJ_1J_2}\epsilon_{jj_1j_2}\epsilon^{J'J_3J_4}\epsilon_{jj_3j_4}q^{-2}_{J_iJ_l}q^{-2}_{j_ij_l}\prod_{k\neq l,i} q^{-1}_{J_kj_k} = \frac{q^2_{JJ'}\operatorname{Tr}(q^{-2})+\delta_{JJ'}}{\det(q)^2} \text{ otherwise.} \tag{4.61}$$

Thus we can finish with a tedious but straightforward computation:

$$\begin{aligned}
\langle \widehat{F}^{J\sigma J'\sigma'}_{\text{der}}\rangle &= \frac{\sigma\sigma'}{4}\frac{1}{4}\left(\frac{8}{3}\right)^4\frac{t^4a^9}{\ell_P^8}(2|\det(q)|^{3/8}t^{[9/8-1]\alpha})^4\times \qquad (4.62)\\
&\quad\times\Big\{[a_1^4\frac{4q^2_{JJ'}}{\det(q)^2}]+\frac{s^2}{4}[4a_1^3([a_1+3a_3]\frac{4q^2_{JJ'}}{\det(q)^2}\operatorname{Tr}(q^{-2})-\frac{a_1}{2}\frac{2[q^2_{JJ'}\operatorname{Tr}(q^{-2})-\delta_{JJ'}]}{\det(q)^2})\\
&\qquad+a_1^2(4[a_1+a_2]^2 6\frac{4q^2_{JJ'}}{\det(q)^2}\operatorname{Tr}(q^{-2})-2a_1[a_1+a_2]12\frac{2[q^2_{JJ'}\operatorname{Tr}(q^{-2})-\delta_{JJ'}]}{\det(q)^2}\\
&\qquad+4a_1^2\frac{q^2_{JJ'}\operatorname{Tr}(q^{-2})+\delta_{JJ'}}{\det(q)^2})]\Big\}\\
&=\frac{\sigma\sigma'}{4}\frac{t^4a^9}{\ell_P^8}\frac{1}{\sqrt{|\det(p)|}}\times\\
&\quad\times\Big\{p^2_{JJ'}+\frac{t}{4}[p^2_{JJ'}\operatorname{Tr}(p^{-2})[4(1+16a_3)-\frac{1}{4}+\frac{8}{3}(3+16a_2)^2-4(3+16a_2)+1]\\
&\qquad+\delta_{JJ'}[\frac{1}{4}+4(3+16a_2)+1]]\Big\}\\
&=\frac{\sigma\sigma'}{4}\frac{t^4a^9}{\ell_P^8}\frac{1}{\sqrt{|\det(p)|}}\Big\{p^2_{JJ'}+t\Big[\frac{1173}{128}p^2_{JJ'}\operatorname{Tr}(p^{-2})+\frac{19}{32}\delta_{JJ'}\Big]\Big\}. \qquad (4.63)
\end{aligned}$$

Again as a last step we transform to the dimensionfull quantity $P=a^2p$:

$$\langle \widehat{F}^{J\sigma J'\sigma'}_{\text{der}}\rangle = \frac{\sigma\sigma'}{4}\frac{1}{\sqrt{|\det(P)|}}\Big\{P^2_{JJ'}+\frac{\ell_P^4}{t}\Big[\frac{1173}{128}P^2_{JJ'}\operatorname{Tr}(P^{-2})+\frac{19}{32}\delta_{JJ'}\Big]\Big\}.$$

4.3.3 The Mass Term:

We now consider the mass term. Its basic building block is the volume operator itself, so we can apply theorem 4.3 with $r=1$. In the by now familiar way we find

$$\begin{aligned}
\langle \widehat{V}_v\rangle &= a^3|det(q)|^{1/2}t^{3\alpha/2}\{1+\frac{s^2}{4}\sum_{M,m}[f^{(2)}_{1/4}(1)C^{Mm,Nn}+f^{(1)}_{1/4}(1)C^{Mm}C^{Nn}\}\\
&= a^3|det(q)|^{1/2}t^{3\alpha/2}\{1+\frac{s^2}{4}\operatorname{Tr}(q^{-2})[\frac{1}{4}-4\frac{3}{32}]\}\\
&= a^3|det(p)|^{1/2}\{1-\frac{t}{32}\operatorname{Tr}(p^{-2})\}\\
&= \sqrt{\det P(v)}\left[1+\frac{\ell_P^7}{\sqrt{t}}\frac{1}{32}\operatorname{Tr}P^{-2}(v)\right]. \qquad (4.64)
\end{aligned}$$

4.3.4 The Maxwell Hamiltonian:

The operators $\widehat{F}_{\text{el}}$ and $\widehat{F}_{\text{mag}}$ differ by their c-number coefficients, but the gravitational operator at the heart of both is the same, corresponding to $N=2$ and $r=1/2$. In both cases we have to compute

$\langle \widehat{q}_{J_1 j}(1/2) \widehat{q}_{J_2 j}(1/2)\rangle$.
Let us use the definitions of a_1, a_2, a_3 given in (4.53) and equations (4.54), (4.55). We find

$$
\begin{aligned}
&< \widehat{q}_{J_1 j}(1/2) \widehat{q}_{J_2 j}(1/2) >= \delta_{j_1 j_2} (2|\det(q)|^{1/4} t^{[3/4-1]\alpha})^2 \times \\
&\qquad \times \Big\{ [\prod_{k=1}^{2} D_{J_k \sigma_k j_k}(1/2)] + \frac{s^2}{4} \sum_{M,m} \Big[\sum_{l=1}^{2} D^{Mm,Mm}_{J_l \sigma_l j_l}(1/2) \prod_{k\neq l} D_{J_k \sigma_k j_k}(1/2) \\
&\qquad + \sum_{1\le i<l\le 2} D^{Mm}_{J_i \sigma_i j_i}(1/2) D^{Mm}_{J_l \sigma_l j_l}(1/2) \prod_{k\neq l,i} D_{J_k \sigma_k j_k}(1/2) \Big] \Big\} \\
&= (2a_1 |\det(q)|^{1/4} t^{[3/4-1]\alpha})^2 \Big\{ q^{-2}_{J_1 J_2} + \frac{s^2}{4} [2([1+3\frac{a_3}{a_1}] q^{-2}_{J_1 J_2} \operatorname{Tr}(q^{-2}) - \frac{1}{2} q^{-4}_{J_1 J_2}) \\
&\qquad + 4[1+\frac{a_2}{a_1}]^2 q^{-2}_{J_1 J_2} \operatorname{Tr}(q^{-2}) - 4[1+\frac{a_2}{a_1}] q^{-4}_{J_1 J_2} + q^{-2}_{J_1 J_2} \operatorname{Tr}(q^{-2})] \Big\} \\
&= (|\det(q)|^{1/4} t^{[3/4-1]\alpha}/4)^2 \{ q^{-2}_{J_1 J_2} + \frac{s^2}{4} [q^{-2}_{J_1 J_2} \operatorname{Tr}(q^{-2})(7 + 3\frac{35}{2^7} - \frac{7}{2} + \frac{3^2 \cdot 5^2}{2^6}) - q^{-4}_{J_1 J_2}(5 - \frac{7}{4})]\} \\
&= \frac{\sqrt{|\det(p)|}}{16} \{ p^{-2}_{J_1 J_2} + t[\frac{763}{512} q^{-2}_{J_1 J_2} \operatorname{Tr}(p^{-2}) - \frac{13}{16} p^{-4}_{J_1 J_2}]\}. \qquad (4.65)
\end{aligned}
$$

We can now employ this result to give the explicit expressions for $\langle \widehat{F}_{\text{el}} \rangle$ and $\langle \widehat{F}_{\text{el}} \rangle$. Upon using the above expectation value, we find that

$$
\begin{aligned}
\langle \widehat{F}^{I\sigma I'\sigma'}_{\text{el}} \rangle &= \frac{\sigma\sigma'}{4} \big[\sqrt{\det P(v)} P^{-2}_{II'} + \frac{\ell_P^4}{t} \Big(\frac{763}{512} P^{-2}_{II'} \operatorname{Tr} P^{-2} - \frac{13}{16} P^{-4}_{II'} \Big) \Big], \\
\langle \widehat{F}^{I\sigma I'\sigma'}_{\text{mag}} \rangle &= \frac{1}{64} \Big[\sqrt{\det P(v)} P^{-2}_{II'} + \frac{\ell_P^4}{t} \Big(\frac{763}{512} P^{-2}_{II'} \operatorname{Tr} P^{-2} - \frac{13}{16} P^{-4}_{II'} \Big) \Big].
\end{aligned}
$$

4.3.5 The Fermionic Hamiltonian

Due to explicit dependence of (4.51) on h'_e the expectation values computed so far are not quite sufficient in order to compute the full expectation value of the fermionic Hamiltonian. Fortunately, the Abelian nature of $U(1)^3$ allows for a simple transcription of theorem 4.1 to this slightly more complicated situation. Notice that at this point coherent states are, for the first time, essential, because weave states, being spin-network states would result in zero expectation values.

Theorem 4.4. *For the classical limit and lowest order quantum corrections of expectation values of monomials of the operators $\hat{q}_{J\sigma j}(r)$ times a holonomy operator for topologically cubic graphs we have*

$$
\begin{aligned}
&\frac{< \psi^t_{\{g_{J\sigma j}\}}, \hat{h}^{\mu}_{J_0 \sigma_0 j_0} \prod_{k=1}^{N} \hat{q}_{J_k \sigma_k j_k}(r_k) \psi^t_{\{g_{J\sigma j}\}} >}{||\psi^t_{\{g_{J\sigma j}\}}||^2} = e^{-t/4} h^{\mu}_{J_0 \sigma_0 j_0} (2|\det(q)|^{r/2} t^{[3r/2-1]\alpha})^N \times \\
&\times \quad \{ [\prod_{k=1}^{N} D_{J_k \sigma_k j_k}(r)] + \frac{s^2}{4} \sum_{M,m} [\sum_{l=1}^{N} D^{Mm,Mm}_{J_l \sigma_l j_l}(r) \prod_{k\neq l} D_{J_k \sigma_k j_k}(r)) \\
&+ \quad \sum_{1\le i<l\le N} D^{Mm}_{J_i \sigma_i j_i}(r) D^{Mm}_{J_l \sigma_l j_l}(r) \prod_{k\neq l,i} D_{J_k \sigma_k j_k}(r)]\}_{p\to p+\mu\sigma_0 \delta_{J_0 j_0} t/4}
\end{aligned}
$$

$$\frac{<\psi^t_{\{g_{J\sigma j}\}}, \prod_{k=1}^N \hat{q}_{J_k\sigma_k j_k}(r_k)\hat{h}^\mu_{J_0\sigma_0 j_0}\psi^t_{\{g_{J\sigma j}\}}>}{||\psi^t_{\{g_{J\sigma j}\}}||^2} = e^{-t/4}h^\mu_{J_0\sigma_0 j_0}(2|\det(q)|^{r/2}t^{[3r/2-1]\alpha})^N \times$$

$$\times \quad \{[\prod_{k=1}^N D_{J_k\sigma_k j_k}(r)] + \frac{s^2}{4}\sum_{M,m}[\sum_{l=1}^N D^{Mm,Mm}_{J_l\sigma_l j_l}(r)\prod_{k\neq l} D_{J_k\sigma_k j_k}(r))$$

$$+ \quad \sum_{1\le i<l\le N} D^{Mm}_{J_i\sigma_i j_i}(r) D^{Mm}_{J_l\sigma_l j_l}(r)\prod_{k\neq l,i} D_{J_k\sigma_k j_k}(r)]\}_{p\to p-\mu\sigma_0\delta_{J_0 j_0}t/4} \tag{4.66}$$

where the constants $D_{J\sigma j}(r), D^{Mm}_{J\sigma j}(r), D^{Mm,Nn}_{J\sigma j}(r)$ are defined in theorem 4.2 while $f^{(k)}_{r/4}(1) = (r/4, k)$ is simply the binomial coefficients. The first correction is small as long as $\alpha < 1/2$. The error as compared to the first quantum correction of order $O(t^{(N[3r/2-1]\alpha}s^2)$ is a constant of order unity times $t^{(N[3r/2-1]\alpha}sT$ and thus small as long as $0 < \alpha$.

Of course, in computing the quantum correction in terms of p or q rather than $p' = p \pm \mu\sigma_0\delta_{J_0 j_0}t/4$ or $q' = p't^{-\alpha}$ up to order t or s^2 respectively one is supposed to insert this substitution into (4.66) and to drop all higher order terms.

Proof of Theorem 4.4:

We begin with the operator identity [**?**]

$$\hat{h}^\mu_{J_0\sigma_0 j_0} = e^{-t/2}e^{-\mu\hat{p}_{J_0\sigma_0 j_0}}\hat{g}^\mu_{J_0\sigma_0 j_0} \tag{4.67}$$

and exploit that our coherent states are eigenstates for $\hat{g}^\mu_{J_0\sigma_0 j_0}$ with eigenvalue $g^\mu_{J_0\sigma_0 j_0}$. Moreover, our coherent states are expanded in terms of momentum operator eigenfunctions on which the operators $e^{-\mu\hat{p}_{J_0\sigma_0 j_0}}$ and $\hat{q}_{J_k\sigma_k j_k}(r_k)$ are simultaneously diagonal. It follows that

$$\begin{aligned}
&< \hat{h}^\mu_{J_0\sigma_0 j_0}\prod_{k=1}^N \hat{q}_{J_k\sigma_k j_k}(r_k) >= e^{-t/2}h^\mu_{J_0\sigma_0 j_0}e^{-\mu p_{J_0\sigma_0 j_0}} < e^{\mu\hat{p}_{J_0\sigma_0 j_0}}\prod_{k=1}^N \hat{q}_{J_k\sigma_k j_k}(r_k) > \\
&< \prod_{k=1}^N \hat{q}_{J_k\sigma_k j_k}(r_k)\hat{h}^\mu_{J_0\sigma_0 j_0} >= e^{-t/2}h^\mu_{J_0\sigma_0 j_0}e^{\mu p_{J_0\sigma_0 j_0}} < e^{-\mu\hat{p}_{J_0\sigma_0 j_0}}\prod_{k=1}^N \hat{q}_{J_k\sigma_k j_k}(r_k) > .
\end{aligned} \tag{4.68}$$

It is therefore sufficient to consider the expectation values

$$\begin{aligned}
& < e^{\nu\hat{p}_{J_0\sigma_0 j_0}}\prod_{k=1}^N \hat{q}_{J_k\sigma_k j_k}(r_k) > \\
= & \frac{\sum_{n_{J\sigma j}} e^{\sum_{J\sigma j}[-tn^2_{J\sigma j}+2n_{J\sigma j}(p_{J\sigma j}+\nu t\delta_{J\sigma j;J_0\sigma_0 j_0}/2)]}\prod_{k=1}^N \lambda^{r_k}_{J_k j_k}(n_{J\sigma j})}{||\psi^t_{\{g_{J\sigma j}\}}||^2} \\
= & \frac{\sum_{n_{J\sigma j}}\int d^{18}x e^{\sum_{J\sigma j}[-x^2_{J\sigma j}+2x_{J\sigma j}(p_{J\sigma j}-i\pi n_{J\sigma j}+\nu t\delta_{J\sigma j;J_0\sigma_0 j_0}/2)/T]}\prod_{k=1}^N \lambda^{r_k}_{J_k j_k}(x_{J\sigma j}/T)}{t^9\ ||\psi^t_{\{g_{J\sigma j}\}}||^2}
\end{aligned} \tag{4.69}$$

where in the second step we have again performed a Poisson transformation with periodicity parameter $T = \sqrt{t}$ (see the companion paper for more details).

As in [**?**] we introduce the coordinates $x^\pm_{Jj} = (x_{J,+,j} \pm x_{J,-,j})/2$ and similar for $p^\pm_{Jj}, n^\pm_{Jj}$ with $x_{Jj} := x^-_{Jj}, p_{Jj} := p^-_{Jj}, n_{Jj} := n^-_{Jj}$. Then one can split the eighteen dimensional integral into two nine dimensional ones with the result

$$
\begin{aligned}
< . > \;&=\; \frac{2^9}{t^9\;||\psi^t_{\{g_{J\sigma j}\}}||^2}\sum_{n_{J\sigma j}}[\int d^9x^+ e^{2\sum_{Jj}[-(x^+_{Jj})^2+2x^+_{Jj}(p^+_{Jj}-i\pi n^+_{Jj}+\nu t\delta_{Jj;J_0j_0}/4)/T]}] \times \\
&\times[\int d^9x e^{2\sum_{Jj}[-x^2_{Jj}+2x_{Jj}(p_{Jj}-i\pi n_{Jj}+\nu\sigma_0 t\delta_{Jj;J_0j_0}/4)/T]}\prod_{k=1}^N \lambda^{r_k}_{J_kj_k}(x_{Jj}/T)]. \qquad (4.70)
\end{aligned}
$$

Then, using the norm of our coherent states and dropping the $O(t^\infty)$ terms which come from the ones with $\sum_{J\sigma j} n^2_{J\sigma j} > 0$ we find

$$
\begin{aligned}
< . > \;&=\; \sqrt{\frac{2}{\pi}}^9\; e^{\frac{2}{t}\sum_{Jj}[(p^++\nu\delta_{J_0j_0}t/4)^2_{Jj}+(p+\nu\sigma_0\delta_{J_0j_0}t/4)^2_{Jj}-(p^+)^2_{Jj}-p^2_{Jj}]} \times \\
&\times[\int d^9x e^{-2\sum_{Jj}x^2_{Jj}}\prod_{k=1}^N\lambda^{r_k}_{J_kj_k}(x+\frac{p+\nu\sigma_0 t\delta_{J_0j_0}/4}{T})] \\
&=\; \sqrt{\frac{2}{\pi}}^9\; e^{\nu p_{J_0\sigma_0 j_0}}e^{t/4}[\int d^9x e^{-2\sum_{Jj}x^2_{Jj}}\prod_{k=1}^N\lambda^{r_k}_{J_kj_k}(x+\frac{p+\nu\sigma_0 t\delta_{J_0j_0}/4}{T})]. \qquad (4.71)
\end{aligned}
$$

Combining (4.68), (4.71) we see that compared to (4.45) and the prefactor of $e^{-t/4}h_{J_0\sigma_0 j_0}$ the remaining integral in (4.71) is the one for the expectation value of the operator monomial $\prod_{k=1}^N \hat{q}_{J_k\sigma_k j_k}(r_k)$ just that we have to evaluate it at $p+\nu\sigma_0 t\delta_{J_0j_0}/4$ instead of at p.
□
We are now ready to apply theorems 4.1 and 4.4 to the case at hand. For $r = 1/2$ we have

$$
a_1 := f^{(1)}_{1/8}(1) = \frac{1}{8},\; a_2 := f^{(2)}_{1/8}(1) = -\frac{1}{8}\frac{7}{16} = -\frac{7}{128},\; a_3 := f^{(3)}_{1/8}(1) = \frac{7}{128}\frac{15}{24} = \frac{35}{1024} \qquad (4.72)
$$

and

$$
\begin{aligned}
D_{J\sigma j}(1/2) &= a_1 q^{-1}_{Jj} \\
D^{Mm}_{J\sigma j}(1/2) &= a_1(2q^{-1}_{Jj}q^{-1}_{Mm} - q^{-1}_{Jm}q^{-1}_{Mj}) + 2a_2 q^{-1}_{Jj}q^{-1}_{Mm} \\
&= 2[a_1+a_2]q^{-1}_{Jj}q^{-1}_{Mm} - a_1 q^{-1}_{Jm}q^{-1}_{Mj} \\
D^{Mm,Nn}_{J\sigma j}(1/2) &= \frac{a_1}{2}[\det(q^{-1})\epsilon_{jmn}\epsilon_{JMN} + q^{-1}_{Jj}(q^{-1}_{Mm}q^{-1}_{Nn} - q^{-1}_{Mn}q^{-1}_{Nm}) \\
&\quad + 2q^{-1}_{Mm}(q^{-1}_{Jj}q^{-1}_{Nn} - q^{-1}_{Jn}q^{-1}_{Nj})] \\
&\quad + 2a_2 q^{-1}_{Jj}(2q^{-1}_{Mm}q^{-1}_{Nn} - q^{-1}_{Mn}q^{-1}_{Nm}) + 3a_3 q^{-1}_{Jj}q^{-1}_{Mm}q^{-1}_{Nn} \\
&= \frac{a_1}{2}\det(q^{-1})\epsilon_{jmn}\epsilon_{JMN} + [\frac{3a_1}{2}+2a_2+3a_3]q^{-1}_{Jj}q^{-1}_{Mm}q^{-1}_{Nn} \\
&\quad - [\frac{a_1}{2}+2a_2]q^{-1}_{Jj}q^{-1}_{Mn}q^{-1}_{Nm} - \frac{a_1}{2}q^{-1}_{Jn}q^{-1}_{Mm}q^{-1}_{Nj}. \qquad (4.73)
\end{aligned}
$$

It follows that

$$
\sum_{M,m} D^{Mm,Mm}_{J\sigma j}(1/2) = [a_1+3a_3]q^{-1}_{Jj}q^{-1}_{Mm}q^{-1}_{Mm} - \frac{a_1}{2}q^{-1}_{Jm}q^{-1}_{Mm}q^{-1}_{Mj}
$$

$$= [a_1 + 3a_3] q^{-1}_{Jj} \mathrm{tr}(q^{-2}) - \frac{a_1}{2} q^{-3}_{Jj}$$

$$\sum_{Mm} D^{Mm}_{J_1\sigma_1 j_1}(1/2) D^{Mm}_{J_2\sigma_2 j_2}(1/2) = (2[a_1+a_2] q^{-1}_{J_1j_1} q^{-1}_{Mm} - a_1 q^{-1}_{J_1m} q^{-1}_{Mj_1})(2[a_1+a_2] q^{-1}_{J_2j_2} q^{-1}_{Mm} - a_1 q^{-1}_{J_2m} q^{-1}_{Mj_2})$$

$$= 4[a_1+a_2]^2 q^{-1}_{J_1j_1} q^{-1}_{J_2j_2} \mathrm{tr}(q^{-2})$$

$$- 2a_1[a_1+a_2](q^{-1}_{J_1j_1} q^{-3}_{J_2j_2} + q^{-1}_{J_2j_2} q^{-3}_{J_1j_1}) + a_1^2 q^{-2}_{J_1J_2} q^{-2}_{j_1j_2}. \quad (4.74)$$

The relevant quantity for the Dirac Hamiltonian is

$$\begin{aligned}
\hat{F}_6 \;&:=\; -\frac{i\hbar}{2} \sum_{v,v' \in V(\gamma)} \{ \\
&\epsilon^{JMN} \epsilon_{jmn} \frac{4^4}{2} [\hat{\theta}_B(v')(\hat{\theta}_A(v))^\dagger - \hat{\theta}'_B(v')(\hat{\theta}'_A(v))^\dagger] \sum_\sigma \times \\
\times \quad & \{< [\delta_{v',e^\sigma_J(v,1)}(\sigma_j[1_2 + \frac{\hat{h}_{J\sigma k}(v) - 1}{2i}\tau_k])_{AB} - \delta_{v',e^\sigma_J(v,0)}\delta_{AB}] \hat{q}_{Mm}(v,1/2)\hat{q}_{Nn}(v,1/2) > \\
& - < \hat{q}_{Mm}(v',1/2)\hat{q}_{Nn}(v',1/2)[\delta_{v,e^\sigma_J(v',1)}([1_2 + \frac{\hat{h}_{J\sigma k}(v')^{-1} - 1}{2i}\tau_k]\sigma_j)_{AB} - \delta_{v,e^\sigma_J(v',0)}\delta_{AB}] >\} \\
+ \quad & 2k_0 \delta_{AB} \delta_{v,v'} [\hat{\theta}'_B(v')(\hat{\theta}_A(v)^\dagger - \hat{\theta}_B(v')(\hat{\theta}'_A(v))^\dagger]\} \qquad (4.75)
\end{aligned}$$

or more explicitly

$$\begin{aligned}
& \hat{F}_6 - \frac{i\hbar}{2} \sum_{v,v' \in V(\gamma)} [-2k_0 \delta_{AB}\delta_{v,v'}[\hat{\theta}'_B(v')(\hat{\theta}_A(v)^\dagger - \hat{\theta}_B(v')(\hat{\theta}'_A(v))^\dagger] \\
= \quad & -\frac{i\hbar}{2} \sum_{v,v' \in V(\gamma)} \epsilon^{JMN} \epsilon_{jmn} \frac{4^4}{2} [\hat{\theta}_B(v')(\hat{\theta}_A(v))^\dagger - \hat{\theta}'_B(v')(\hat{\theta}'_A(v))^\dagger] \sum_\sigma \times \\
\times \quad & \{[(\delta_{v',e^\sigma_J(v,1)}(\sigma_j[1_2 + \frac{(-1)}{2i}\tau_k])_{AB} - \delta_{v',e^\sigma_J(v,0)}\delta_{AB}) < \hat{q}_{Mm}(v,1/2)\hat{q}_{Nn}(v,1/2) > \\
& +(\delta_{v',e^\sigma_J(v,1)}(\sigma_j \frac{1}{2i}\tau_k])_{AB} < \hat{h}_{J\sigma k}(v)\hat{q}_{Mm}(v,1/2)\hat{q}_{Nn}(v,1/2) >] \\
& -[(\delta_{v,e^\sigma_J(v',1)}([1_2 + \frac{(-1)}{2i}\tau_k]\sigma_j)_{AB} - \delta_{v,e^\sigma_J(v',0)}\delta_{AB}) < \hat{q}_{Mm}(v',1/2)\hat{q}_{Nn}(v',1/2) > \\
& +\delta_{v,e^\sigma_J(v',1)}(\frac{1}{2i}\tau_k\sigma_j)_{AB} < \hat{q}_{Mm}(v',1/2)\hat{q}_{Nn}(v',1/2)\hat{h}_{J\sigma k}(v')^{-1} >]\}. \qquad (4.76)
\end{aligned}$$

Let us write $q = p(v)t^{-\alpha}$, $q' = p(v')t^{-\alpha}$, $q_1 = p_1(v)t^{-\alpha}, q'_1 = p_1(v')t^{-\alpha}$ where $p_1(v) = p(v) + \sigma t\delta_{Jk}/4$, $p_1(v') = p(v') - \sigma t\delta_{Jk}/4$. Then, using theorem 4.4, we may write (4.76) in the reduced form

$$\begin{aligned}
& \hat{H}^{eff}_\gamma - \frac{i\hbar}{2} \sum_{v,v' \in V(\gamma)} [-2k_0 \delta_{AB}\delta_{v,v'}[\hat{\theta}'_B(v')(\hat{\theta}_A(v)^\dagger - \hat{\theta}_B(v')(\hat{\theta}'_A(v))^\dagger] \\
= \quad & -\frac{i\hbar}{2} \sum_{v,v' \in V(\gamma)} \epsilon^{JMN} \epsilon_{jmn} \frac{4^4}{2} [\hat{\theta}_B(v')(\hat{\theta}_A(v))^\dagger - \hat{\theta}'_B(v')(\hat{\theta}'_A(v))^\dagger] \sum_\sigma \times \\
\times \quad & \{[(\delta_{v',e^\sigma_J(v,1)}(\sigma_j[1_2 + \frac{(-1)}{2i}\tau_k])_{AB} - \delta_{v',e^\sigma_J(v,0)}\delta_{AB}) < \hat{q}_{Mm}(1/2)\hat{q}_{Nn}(1/2) >_q
\end{aligned}$$

$$
\begin{aligned}
&+e^{-t/4}(\delta_{v',e^\sigma_J(v,1)}(\sigma_j\frac{h_{J\sigma k}(v)}{2i}\tau_k])_{AB} < \hat{q}_{Mm}(1/2)\hat{q}_{Nn}(1/2) >_{q_1}] \\
&-[(\delta_{v,e^\sigma_J(v',1)}([1_2+\frac{(-1)}{2i}\tau_k]\sigma_j)_{AB} - \delta_{v,e^\sigma_J(v',0)}\delta_{AB}) < \hat{q}_{Mm}(1/2)\hat{q}_{Nn}(1/2) >_{q'} \\
&+e^{-t/4}\delta_{v,e^\sigma_J(v',1)}(\frac{h_{J\sigma k}(v')^{-1}}{2i}\tau_k\sigma_j)_{AB} < \hat{q}_{Mm}(1/2)\hat{q}_{Nn}(1/2) >_{q_1'}]\}.
\end{aligned} \tag{4.77}
$$

It remains to apply theorem 4.4. We have explicitly for arbitrary invertible q

$$
\begin{aligned}
& \epsilon^{JJ_1J_2}\epsilon^{jj_1j_2} < \hat{q}_{J_1j_1}(1/2)\hat{q}_{J_2j_1}(1/2) > \equiv \epsilon^{JJ_1J_2}\epsilon^{jj_1j_2} < \hat{q}_{J_1\sigma_1j_1}(1/2)\hat{q}_{J_2\sigma_2j_2}(1/2) > \\
= \; & \epsilon^{JJ_1J_2}\epsilon^{jj_1j_2}(2|\det(q)|^{1/4}t^{[3/4-1]\alpha})^2 \times \\
\times \; & \{[\prod_{k=1}^{2} D_{J_k\sigma_kj_k}(1/2)] + \frac{s^2}{4}\sum_{M,m}[\sum_{l=1}^{2} D^{Mm,Mm}_{J_l\sigma_lj_l}(1/2)\prod_{k\neq l} D_{J_k\sigma_kj_k}(1/2)) \\
+ \; & \sum_{1\le i<l\le 2} D^{Mm}_{J_i\sigma_ij_i}(1/2) D^{Mm}_{J_l\sigma_lj_l}(1/2)\prod_{k\neq l,i} D_{J_k\sigma_kj_k}(1/2)]\} \\
= \; & \epsilon^{JJ_1J_2}\epsilon^{jj_1j_2}(2|\det(q)|^{1/4}t^{[3/4-1]\alpha})^2 \times \\
\times \; & \{a_1^2 q^{-1}_{J_1j_1}q^{-1}_{J_2j_2} + \frac{s^2}{4}[a_1(q^{-1}_{J_1j_1}([a_1+3a_3]q^{-1}_{J_2j_2}\mathrm{tr}(q^{-2}) - \frac{a_1}{2}q^{-3}_{J_2j_2}) + q^{-1}_{J_2j_2}([a_1+3a_3]q^{-1}_{J_1j_1}\mathrm{tr}(q^{-2}) - \frac{a_1}{2}q^{-3}_{J_1j_1})) \\
+ \; & 4[a_1+a_2]^2 q^{-1}_{J_1j_1}q^{-1}_{J_2j_2}\mathrm{tr}(q^{-2}) - 2a_1[a_1+a_2](q^{-1}_{J_1j_1}q^{-3}_{J_2j_2} + q^{-1}_{J_2j_2}q^{-3}_{J_1j_1}) + a_1^2 q^{-2}_{J_1J_2}q^{-2}_{j_1j_2}]\}.
\end{aligned} \tag{4.78}
$$

We have

$$
\begin{aligned}
\epsilon^{JJ_1J_2}\epsilon^{jj_1j_2}q^{-1}_{J_1j_1}q^{-1}_{J_2j_2} &= 2\det(q^{-1})q_{Jj} \\
\epsilon^{JJ_1J_2}\epsilon^{jj_1j_2}q^{-1}_{J_1j_1}q^{-3}_{J_2j_2} &= \det(q^{-1})(q_{Jj}\mathrm{tr}(q^{-2} - q^{-1}_{Jj}) \\
\epsilon^{JJ_1J_2}\epsilon^{jj_1j_2}q^{-2}_{j_1j_2} &= 0.
\end{aligned} \tag{4.79}
$$

Thus we can finish (4.78) with

$$
\begin{aligned}
& \epsilon^{JJ_1J_2}\epsilon^{jj_1j_2} < \hat{q}_{J_1j_1}(1/2)\hat{q}_{J_2j_1}(1/2) > \\
= \; & (2|\det(q)|^{1/4}t^{[3/4-1]\alpha})^2 \det(q^{-1}) \times \\
\times \; & \{2a_1^2 q_{Jj} + \frac{s^2}{4}[2a_1(2[a_1+3a_3]q_{Jj}\mathrm{tr}(q^{-2}) - \frac{a_1}{2}[q_{Jj}\mathrm{tr}(q^{-2}) - q^{-1}_{Jj}]) \\
+ \; & 8[a_1+a_2]^2 q_{Jj}\mathrm{tr}(q^{-2}) - 4a_1[a_1+a_2][q_{Jj}\mathrm{tr}(q^{-2}) - q^{-1}_{Jj}]]\} \\
= \; & 4t^{-\alpha/2}/\sqrt{\det(q)}\{2a_1^2 q_{Jj} \\
& +\frac{s^2}{4}[(4a_1[a_1+3a_3] + 8[a_1+a_2]^2)q_{Jj}\mathrm{tr}(q^{-2}) - (a_1^2 + 4a_1[a_1+a_2])(q_{Jj}\mathrm{tr}(q^{-2}) - q^{-1}_{Jj})]\} \\
= \; & \frac{t^{-\alpha/2}}{16\sqrt{\det(q)}}\{2q_{Jj} \\
& +\frac{s^2}{4}[(4[1+24a_3] + 8[1+8a_2]^2)q_{Jj}\mathrm{tr}(q^{-2}) - (1+4[1+8a_2])(q_{Jj}\mathrm{tr}(q^{-2}) - q^{-1}_{Jj})]\} \\
= \; & \frac{t^{-\alpha/2}}{16\sqrt{\det(q)}}\{2q_{Jj} + \frac{s^2}{4}[(4+\frac{3^2\cdot 5}{2^5}] + \frac{3^4}{2^5})q_{Jj}\mathrm{tr}(q^{-2}) - \frac{13}{4}(q_{Jj}\mathrm{tr}(q^{-2}) - q^{-1}_{Jj})]\}
\end{aligned}
$$

$$
\begin{aligned}
&= \frac{1}{8\sqrt{\det(p)}}\{p_{Jj} + \frac{t}{2^7}[75 p_{Jj}\mathrm{tr}(p^{-2}) + 52 p^{-1}_{Jj})]\} \\
&= \frac{t^{-\alpha/2}}{8\sqrt{\det(q)}}\{q_{Jj} + \frac{s^2}{2^7}[75 q_{Jj}\mathrm{tr}(q^{-2}) + 52 q^{-1}_{Jj})]\}.
\end{aligned} \tag{4.80}
$$

Notice that the classical limit is *precisely* the correct one while the relative first quantum correction is given by approximately $1.0 s^2 \delta_{Jj}$ for flat initial data.

Now we should compute the additional corrections arising when expanding

$$
e^{-t/4}\epsilon^{JJ_1J_2}\epsilon^{jj_1j_2} < \hat{q}_{J_1j_1}(1/2)\hat{q}_{J_2j_1}(1/2) >_{q\to q+t^{1-\alpha}\nu\sigma_0\delta_{J_0j_0}/4} \tag{4.81}
$$

at q up to order s^2. However, it is clear that the the additional correction in $e^{-t/4}-1 = s^2 O(t^{2\alpha})$ and the one from $\delta q = s^2 O(t^\alpha)$ are both of higher order in s so that we can drop the factors of $e^{-t/4}$ and the substitutions $q \to q_1, q' \to q'_1$ in (4.77) which therefore can be written, up to order s^2 as

$$
\begin{aligned}
& \hat{H}^{eff}_\gamma - \frac{i\hbar}{2}\sum_{v,v'\in V(\gamma)} [-2k_0\delta_{AB}\delta_{v,v'}[\hat{\theta}'_B(v')(\hat{\theta}_A(v)^\dagger - \hat{\theta}_B(v')(\hat{\theta}'_A(v))^\dagger] \\
= & -\frac{i\hbar}{2}\sum_{v,v'\in V(\gamma)} \epsilon^{JMN}\epsilon^{jmn}[\hat{\theta}_B(v')(\hat{\theta}_A(v))^\dagger - \hat{\theta}'_B(v')(\hat{\theta}'_A(v))^\dagger]\sum_\sigma \times \\
\times & \{[(\delta_{v',e^\sigma_J(v,1)}(\sigma_j[1_2 + \frac{h_{J\sigma k}(v)-1}{2i}\tau_k])_{AB} - \delta_{v',e^\sigma_J(v,0)}\delta_{AB}) < \hat{q}_{Mm}(1/2)\hat{q}_{Nn}(1/2) >_{q=q(v)}] \\
& -[(\delta_{v,e^\sigma_J(v',1)}([1_2 + \frac{h_{J\sigma k}(v')^{-1}-1}{2i}\tau_k]\sigma_j)_{AB} - \delta_{v,e^\sigma_J(v',0)}\delta_{AB}) < \hat{q}_{Mm}(1/2)\hat{q}_{Nn}(1/2) >_{q=q(v')}]\} \\
= & -\frac{i\hbar}{2}\sum_{v,v'\in V(\gamma)} [\hat{\theta}_B(v')(\hat{\theta}_A(v))^\dagger - \hat{\theta}'_B(v')(\hat{\theta}'_A(v))^\dagger]\sum_\sigma \times \\
\times & \{[(\delta_{v',e^\sigma_J(v,1)}(\sigma_j[1_2 + \frac{h_{J\sigma k}(v)-1}{2i}\tau_k])_{AB} - \delta_{v',e^\sigma_J(v,0)}\delta_{AB})\frac{1}{8\sqrt{\det(p)}} \times \\
& \times(p_{Jj} + \frac{t}{2^7}[75 p_{Jj}\mathrm{tr}(p^{-2}) + 52 p^{-1}_{Jj})])(v)] \\
& -[(\delta_{v,e^\sigma_J(v',1)}([1_2 + \frac{h_{J\sigma k}(v')^{-1}-1}{2i}\tau_k]\sigma_j)_{AB} - \delta_{v,e^\sigma_J(v',0)}\delta_{AB})\frac{1}{8\sqrt{\det(p)}} \times \\
& \times(p_{Jj} + \frac{t}{2^7}[75 p_{Jj}\mathrm{tr}(p^{-2}) + 52 p^{-1}_{Jj})])(v')]\}.
\end{aligned} \tag{4.82}
$$

5 Towards Dispersion Relations

In the present section, we will bring together some of the results of the companion paper and the previous section: We will compute corrections to the standard dispersion relations for the scalar and the electromagnetic field resulting from its coupling to QGR. The necessary calculations are performed in section 5.1 for the scalar and in 5.2 for the electromagnetic field. Similar computations can be performed for the fermions but they give no new insights so that we leave this to the interested reader. We have set up the problem in such a way that the calculations for an arbitrary background metric but for a start we confine ourselves to the flat one. In section 5.3 we will discuss the results and compare them to those obtained in [?, ?]. In our companion paper we have given some conceptual discussion of the issues involved in obtaining dispersion relation from QGR, so we will mainly focus on the concrete calculations.

In [?] we have obtained Hamiltonian operators for the matter fields of the form

$$\widehat{H}^{\rm eff}_\gamma = \frac{1}{2}\sum_{v,v',l,l'} \widehat{p}_l(v)P^{ll'}(v,v')\widehat{p}_{l'}(v') + \widehat{q}_l(v)Q^{ll'}(v,v')\widehat{q}_{l'}(v'),$$

where the coefficients P, Q are the expectation values of specific operators on the gravitational Hilbert space. We have computed these expectation values in the preceeding section.

Note that these Hamiltonians are normal ordered with respect to the annihilation and creation operators defined in [?]. Thus, the expectation value of these Hamiltonians in a coherent state peaked at a specific classical field configuration will yield *precisely* its classical value. Therefore, in discussing the dispersion relations, we will assume the matter quantum fields to be in a coherent state and can effectively work with the classical fields p, q. A similar argument can be given for the fully quantized Hamiltonians of [?], only that one has to consider a coherent state for the *combined* system of quantum matter *and* quantum gravity displayed in section 4 of [?] as well.

Summing up, in the following we will investigate Hamiltonians of the form

$$\langle \widehat{H}^{\rm eff}_\gamma \rangle = \frac{1}{2}\sum_{v,v',l,l'} p_l(v)P^{ll'}(v,v')p_{l'}(v') + q_l(v)Q^{ll'}(v,v')q_{l'}(v'). \tag{5.1}$$

The coefficients P, Q can in principle be taken to be expectation values in a coherent state for the gravitational field peaked at an arbitrary point of the classical phase space. However, since we are interested in dispersion relations, a notion that by definition describes the propagation of fields in flat space, we will restrict considerations to the case of GCS approximating flat Euclidean space (denoted by $\Psi_{\rm flat}$ in the following. Also, when considering application to situations such as the γ-ray burst effect, the curvature radius is always huge compared to Planck length and does therefore not lead to any new quantum effects but just to classical redshifts which can easily be accounted for.

Let us choose the canonical Euclidean coordinate system as global coordinates on Σ. In the $\mathrm{U}(1)^3$-setting, we can model the flat space situation by choosing the classical values

$$A^I_a(x) = 0, \qquad E^a_I(x) = \delta^a_I \qquad \text{for all } x \in \Sigma$$

with respect to our global coordinates. Therefore all holonomies are trivial and for the fluxes we find

$$p^e_i(v) = \frac{1}{a^2}\int_{S_e} dn^i.$$

We will also use the dimensionfull quantity $P^e_i(v) = a^2 p^e_i(v)$.

Let us come back to the discussion of (5.1): Since the coefficients in these Hamiltonians vary from vertex to vertex, the equations of motion induced by (5.1) are still highly complicated and an exact analytical treatment is beyond the scope of the present paper. Moreover, the solutions to the equations of motion will not have the character of plane waves, so the notion of a dispersion relation is ill defined anyway.

In [?] we argued that in the limit of low energies or, equivalently, large wavelength the field propagation induced by (5.1) *can* be described by a dispersion relation: The graph γ, the GCS is based on, breaks Euclidean invariance. However on large scales this invariance is approximately restored.

As we can not easily (5.1) compute the solutions to the equations of motion of (5.1) and show that they reduce to approximate plane waves with a specific dispersion relation in the low energy limit, the question is how one can nevertheless obtain the dispersion relation governing the propagation for low energies.

In [?] we have sketched a tentative answer, which we will work out in the present section for the examples of the scalar and the electromagnetic field. Let us review the basic idea of the procedure before we spell out the details: We are going to replace (5.1) by a simpler Hamiltonian which

- is a good approximation of (5.1) for slowly varying q and p and
- is simple enough such that the EOM can be solved exactly.

The resulting theory will be an approximation for low energies, the detailed information contained in the full Hamiltonian (5.1) which is only relevant for processes of very high energy gets integrated out. This idea underlies also the works [?] and [?] and, at a rather simple level, is the basis for the recovery of continuum elasticity theory from the atomic description in solid state physics (see for example [?]).

We will now turn to the scalar field Hamiltonian and explain the steps we will take to implement the above idea in detail. The Maxwell Hamiltonian will be treated along the same lines in section 5.2.

5.1 Dispersion Relation for the Scalar Field

The basic field variables underlying the quantization of the scalar field in [?],

$$\phi_v = \phi(\vec{x}(v)), \qquad \text{and} \qquad \pi_v = \int_{R_v} \pi, \tag{5.2}$$

were represented by the operators $-i \ln U(v)$, Y_v. R_v is the cell containing v in a polyhedral decomposition of Σ dual to γ. According to what we have said in the introduction to this chapter, in the considerations to follow we will replace these operators by their classical counterparts (5.2) upon assuming the quantum fields to be in a coherent state.

Using the results of section 4, the Hamiltonian for the scalar field we are considering can be written as

$$H^{\text{eff}}_{\text{KG}} = \frac{1}{2Q_{\text{KG}}} \left(F_{\text{kin}}(\pi) + F_{\text{der}}(\phi) + K^2 F_{\text{m}} \right), \tag{5.3}$$

where

$$\begin{aligned} F_{\text{m}} &= \sum_v \sqrt{\det P(v)} \left[1 + \frac{\ell_P^7}{\sqrt{t}} \frac{1}{32} \operatorname{Tr} P^{-2}(v) \right] \phi_v^2, \\ F_{\text{der}} &= \frac{1}{4} \sum_v \sum_{I\sigma I'\sigma'} \left[\frac{\sigma\sigma' P^2_{II'}(v)}{\sqrt{\det P(v)}} \right. \\ &\qquad \left. + \frac{\ell_P^4}{t} \frac{\sigma\sigma'}{\sqrt{\det P(v)}} \left(\frac{1173}{128} \operatorname{Tr}(P^{-2}) P^2_{II'}(v) + \frac{19}{32} \delta_{II'} \right) \right] \partial^+_{e_{\sigma I}} \phi_v \partial^+_{e_{\sigma' I'}} \phi_v, \end{aligned}$$

$$F_{\rm kin} = \sum_v \frac{1}{\sqrt{\det P(v)}} \left[1 + \frac{\ell_P^4}{t} \frac{1707}{512} \operatorname{Tr} P^{-2}(v) \right] \pi_v^2.$$

Now we will express the field quantities ϕ_v, π_v by the basic fields $\phi(\vec{x}), \pi(\vec{x})$, using an approximation which is good in the case $\phi(\vec{x}), \pi(\vec{x})$ vary only very little on the scale ϵ of the graph. The idea is to isolate the rough structural properties of (5.3) that lead to corrections as compared to the standard dispersion relations and to discard the microscopic details that will only yield higher order corrections which are not visible in the long wavelength regime.

To this end, we Taylor expand the field variables ϕ_v, π_v around the location $\vec{x}(v)$ of the vertex v, i.e. we make the replacements

$$\begin{aligned} \phi_v &\longrightarrow \phi(\vec{x}(v)), \\ \pi_v &\longrightarrow \pi(\vec{x}(v)) \mathrm{Vol}(R_v) + a^{(a)}(v) \partial_a \pi(\vec{x}(v)) + \ldots, \\ \partial^+_{e_I} \phi_v &\longrightarrow b_I^{(a)} \partial_a \phi(\vec{x}(v)) + b_I^{(a)} b_I^{(a')} \partial_a \partial_{a'} \phi(\vec{x}(v)) + \ldots. \end{aligned}$$

and truncate the right hand sides at the desired order. Note that in the above formulae we have introduced the geometric quantities

$$b_I^{(a)}(v) \doteq x^a(f(e_I(v))) - x^a(v), \qquad a^{(a)}(v) \doteq \int_{R_v} x^a \, d^3x,$$

and let us furthermore define

$$\widetilde{b}_I^{(a)}(v) \doteq \frac{1}{2} \left(x^a[f(e_I(v))] - x^a[f(e_I^-(v))] \right)$$

which we will have opportunity to use below. Also, it is perhaps worthwhile to remind the reader at this point that all edges are taken to be outgoing from v.

Then we replace the coefficients of the continuum fields by graph averages and the sums by integrals. As argued in [**?**], this is a good approximation, as long as ϕ and π are slowly varying on the graph scale ϵ. Let us detail this step for the example of the mass term. We write

$$\begin{aligned} F_{\rm m} &= \sum_v \mathrm{Vol}(R_v) \frac{\sqrt{\det P(v)}}{\mathrm{Vol}(R_v)} \left[1 + \frac{\ell_P^7}{\sqrt{t}} \frac{1}{32} \operatorname{Tr} P^{-2}(v) \right] \phi_v^2 \\ &\approx \sum_v \phi_v^2 \mathrm{Vol}(R_v) \left(\langle\langle \frac{\sqrt{\det P(\,\cdot\,)}}{\mathrm{Vol}(R_\cdot)} \rangle\rangle + \frac{\ell_P^7}{\sqrt{t}} \frac{1}{32} \langle\langle \frac{\sqrt{\det P(\,\cdot\,)} \operatorname{Tr} P^{-2}(\,\cdot\,)}{\mathrm{Vol}(R_\cdot)} \rangle\rangle \right) \\ &\approx \int_\Sigma \phi(x) \, d^3x \; \left(\langle\langle \frac{\sqrt{\det P(\,\cdot\,)}}{\mathrm{Vol}(R_\cdot)} \rangle\rangle + \frac{\ell_P^7}{\sqrt{t}} \frac{1}{32} \langle\langle \frac{\sqrt{\det P(\,\cdot\,)} \operatorname{Tr} P^{-2}(\,\cdot\,)}{\mathrm{Vol}(R_\cdot)} \rangle\rangle \right) \end{aligned}$$

where $\langle\langle \cdot \rangle\rangle$ denotes the *graph average*

$$\langle\langle C(\,\cdot\,) \rangle\rangle \doteq \frac{1}{N} \sum_v C(v)$$

for vertex dependent quantities $C(v)$. N denotes the number of vertices of the graph. In case the graph has a countably infinite number of vertices, the above definition has to be replaced by an appropriate limit of finite sums.

Analogously, we make the replacements

$$
\begin{aligned}
F_{\mathrm{kin}} &\longrightarrow \int_\Sigma \left(A_0 + A_1\right) \pi^2(\vec{x}) + A_0^{(a)(a')} \partial_a \pi(\vec{x}) \partial_{a'} \pi(\vec{x}) + \ldots \; d^3x, \\
F_{\mathrm{der}} &\longrightarrow \int_\Sigma \left(B_0^{(a)(a')} + B_1^{(a)(a')} \right) \partial_a \phi(\vec{x}) \partial_{a'} \phi(\vec{x}) \\
&\qquad\qquad + \frac{1}{4} B_0^{(ab)(a'b')} \partial_a \partial_b \phi(\vec{x}) \partial_{a'} \partial_{b'} \phi(\vec{x}) + \frac{1}{3} B_0^{(abc)(a')} \partial_a \partial_b \partial_c \phi(\vec{x}) \partial_{a'} \phi(\vec{x}) + \ldots \; d^3x, \\
F_{\mathrm{m}} &\longrightarrow \int_\Sigma \left(C_0 + C_1\right) \phi^2(\vec{x}) + \ldots \; d^3x
\end{aligned}
$$

where the coefficients in the kinetic term are defined to be

$$
\begin{aligned}
A_0 &= \langle\langle \frac{V_v^2}{\sqrt{\det P(v)}} \rangle\rangle, \\
A_1 &= \frac{1707}{512} \frac{\ell_P^4}{t} \langle\langle \frac{V^2 \operatorname{Tr} P(v)}{\sqrt{\det P(v)}} \rangle\rangle, \\
A_1^{(a)(a')} &= \langle\langle \frac{a^{(a)}(v) a^{(a')}(v)}{\sqrt{\det P(v)}} \rangle\rangle,
\end{aligned}
$$

the ones in the derivative term as

$$
\begin{aligned}
B_0^{(a)(a')} &= \sum_{I,I'} \langle\langle \sqrt{\det P(v)} P_{II'}^2 \widetilde{b}_I^a \widetilde{b}_{I'}^{a'}(v) \rangle\rangle, \\
B_1^{(a)(a')} &= \frac{\ell_P^4}{t} \sum_{I,I'} \langle\langle \sqrt{\det P(v)} \left(\frac{1173}{128} \operatorname{Tr}(P^{-2}) P_{II'}^2(v) + \frac{19}{32} \delta_{II'} \right) \widetilde{b}_I^a \widetilde{b}_{I'}^{a'} \rangle\rangle, \\
B_0^{(abc)(a')} &= \sum_{I,I'} \langle\langle \sqrt{\det P(v)} P_{II'}^2 \widetilde{b}_I^a \widetilde{b}_I^b \widetilde{b}_I^c \widetilde{b}_{I'}^{a'}(v) \rangle\rangle, \\
B_0^{(ab)(a'b')} &= \sum_{I,I'} \langle\langle \sqrt{\det P(v)} P_{II'}^2 \widetilde{b}_I^a \widetilde{b}_I^b \widetilde{b}_{I'}^{a'} \widetilde{b}_{I'}^{b'}(v) \rangle\rangle.
\end{aligned}
$$

and finally the coefficients in the mass term by

$$
\begin{aligned}
C_0 &\doteq \langle\langle \sqrt{\det P(v)} \rangle\rangle, \\
C_1 &\doteq \frac{1}{32} \frac{\ell_P^7}{\sqrt{t}} \langle\langle \sqrt{\det P(v)} \operatorname{Tr} P^{-2}(v) \rangle\rangle.
\end{aligned}
$$

Note that we have just written down the leading order terms and the first order corrections, where a "first order correction" is either of

1. A term that is *next to leading order* in the Taylor expansion and *leading* order with respect to the fluctuation calculation.
2. A term that is *leading order* in the Taylor expansion and next to leading order in the fluctuation calculation.

Terms that are leading order in the fluctuation calculation carry a superscript 0 while that are first order corrections in the fluctuation calculation are marked by a superscript 1. Finally note that we have dropped terms that end up to be a total derivative and therefore do not contribute to the Hamiltonian.

Before we write down the resulting dispersion relation we invoke our restriction to random processes which imply Euclidean invariance on large scales of the resulting random graphs. That is we assume

$$A_0^{(a)(a')} \sim \delta^{aa'}, \qquad B_0^{(a)(a')} \sim \delta^{aa'}, \qquad B_1^{(a)(a')} \sim \delta^{aa'}.$$

For the tensors of fourth rank, the situation is slightly more complicated: $\delta^{ab}\delta^{cd}$, $\delta^{ac}\delta^{db}$ and $\delta^{ad}\delta^{bc}$ span the space of rotationally invariant tensors of fourth rank. But contraction of any of them with $k_a k_b k_c k_d$ is equal to $|k|^4$.

We can now write down the dispersion relation for the Hamiltonian resulting from the above replacements

$$\begin{aligned} \omega^2(\vec{k}) =& K^2\left[A_0C_0 + A_0C_1 + A_1C_0\right] \\ &+ |k|^2\left[A_0B_0 + A_0B_1^{(1)(1)} + A_1B_0^{(1)(1)} + K^2A_0^{(1)(1)}C_0\right] \\ &+ |k|^4\left[\frac{1}{4}A_0B_0^{(11)(11)} - \frac{1}{3}A_0B_0^{(111)(1)} + A_0^{(1)(1)}B_0^{(1)(1)}\right] \\ &+ \ldots . \end{aligned} \tag{5.4}$$

We will discuss the physical content of (5.4) in section 5.3. Before that, we give a similar calculation for the electromagnetic field.

5.2 The Electromagnetic Field

This section is devoted to the calculation of an (approximate) dispersion relation for the electromagnetic field. The treatment is completely analogous to the one given for the scalar field in the last section, so we can be rather brief, here. Again, we introduce the continuum fields $A_a(\vec{x})$, $E^a(\vec{x})$ underlying the regularization and quantization of the Hamiltonian performed in [?] and also the classical quantities

$$\begin{aligned} E_e &= \int_{S_e} E \text{ which was represented by } Y_e, \\ A_e &= \int_e A \text{ which was represented by } -i\ln H_e \end{aligned} \tag{5.5}$$

(subject to the subtleties associated with the logarithm spelled out in detail in [?]) in the quantum Hamiltonian (4.50). When we replace the gravitational operators in the Hamiltonian by their expectation values obtained in the last section and the operators for the matter fields by their classical counterparts (5.5) we get

$$\langle \widehat{H}_{M,\gamma}\rangle_{\Psi_{\text{flat}}} = \frac{1}{2Q_{\text{M}}}\left(F_{\text{el}}(E) + F_{\text{mag}}(B)\right)$$

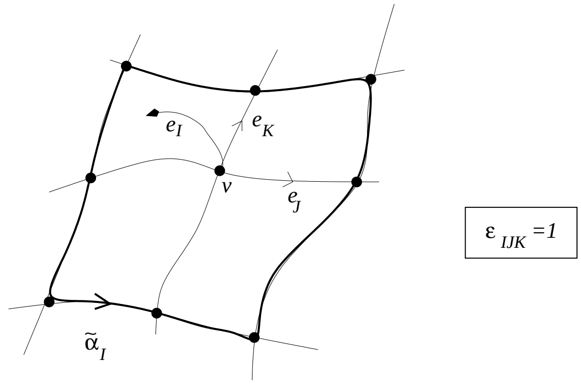

Figure 1: The loop $\widetilde{\alpha}_I(v)$

where

$$F_{\mathrm{el}}(E) = \sum_v \sum_{I\sigma I'\sigma'} \left[\sqrt{\det P(v)} P^{-2}_{II'} + \frac{\ell_P^4}{t}\Big(\frac{763}{512} P^{-2}_{II'} \operatorname{Tr} P^{-2} - \frac{13}{16} P^{-4}_{II'}\Big)\right] \frac{\sigma\sigma'}{4} E_{e_{\sigma I}}(v) E_{e_{\sigma' I'}}(v),$$

$$F_{\mathrm{mag}}(A) = \frac{1}{16} \sum_v \sum_{II'} \left[\sqrt{\det P(v)} P^{-2}_{II'} + \frac{\ell_P^4}{t}\Big(\frac{763}{512} P^{-2}_{II'} \operatorname{Tr} P^{-2} - \frac{13}{16} P^{-4}_{II'}\Big)\right] A_{\widetilde{\alpha}_I} A_{\widetilde{\alpha}_{I'}},$$

where $\widetilde{\alpha}(v)$ is the loop around the vertex v "in the I-plane" as depicted in figure 1. Now we Taylor-expand the A_α, E_e. To this end we introduce some geometric quantities:

$$s^e_a(v) \doteq \int_{S^e_v} n_a(\vec{y})\, dy, \qquad s^e_{ab}(v) \doteq \int_{S^e_v} n_a(\vec{y})(\vec{y} - \vec{x}(v))_b\, dy,$$
$$s^e_{abc}(v) \doteq \int_{S^e_v} n_a(\vec{y})(\vec{y} - \vec{x}(v))_b(\vec{y} - \vec{x}(v))_c\, dy,$$

where n denotes the normal to the surface of integration. Moreover

$$\widetilde{s}^I_a(v) \doteq \frac{1}{2}\left(s^{e_I}_b(v) - s^{e_I^-}_c(v)\right), \qquad \widetilde{s}^I_{ab}(v) \doteq \frac{1}{2}\left(s^{e_I}_{ab}(v) - s^{e_I^-}_{ab}(v)\right), \ldots.$$

Now we can make the replacement for the electric field:

$$E_e \longrightarrow s^e_a(v) E^a(\vec{x}(v)) + s^e_{ab}(v)\partial^b E^a(\vec{x}(v)) + \ldots .$$

We proceed in a similar fashion for the connection:

$$b^{ab}_\alpha(v) \doteq \int_0^1 \dot{\alpha}^a(s)(\vec{\alpha}(s) - \vec{x}(v))^b, \qquad b^{abc}_\alpha(v) \doteq \int_0^1 \dot{\alpha}^a(s)(\vec{\alpha}(s) - \vec{x}(v))^b(\vec{\alpha}(s) - \vec{x}(v))^c \tag{5.6}$$

whence we replace

$$A_\alpha \longrightarrow b^{ab}_\alpha(v)\partial_b A_a(\vec{x}(v)) + \frac{1}{2} b^{abc}_\alpha(v)\partial_b\partial_c A_a(\vec{x}(v)) + \ldots .$$

Inserting this into the expressions for $F_{\rm el}$ and $F_{\rm mag}$ and subsequently replacing the resulting coefficients by graph averages results in the total replacement

$$\begin{aligned}
F_{\rm el} &\longrightarrow \int_\Sigma \left(S^{(0)}_{(a)(a')} + S^{(1)}_{(a)(a')}\right) E^a(\vec{x})E^{a'}(\vec{x}) + 2S^{(0)}_{(a)(a'b')}E^a(\vec{x})\partial^{b'}E^{a'}(\vec{x}) + \ldots\,,\\
F_{\rm mag} &\longrightarrow \left(B_0^{(ab)(a'b')} + B_1^{(ab)(a'b')}\right)\partial_b A_a(\vec{x})\partial_{b'}A_{a'}(\vec{x}) + B_0^{(ab)(a'b'c')}\partial_b A_a(\vec{x})\partial_{b'}\partial_{c'}A_{a'}(\vec{x}) + \ldots\,,
\end{aligned}\tag{5.7}$$

where

$$\begin{aligned}
S^{(0)}_{(a)(a')} &= \sum_{II'}\langle\langle\sqrt{\det P}P^{-2}_{II'}(\,\cdot\,)\widetilde{s}^I_a(\,\cdot\,)\widetilde{s}^{I'}_{a'}(\,\cdot\,)\rangle\rangle,\\
S^{(1)}_{(a)(a')} &= \frac{\ell_P^4}{t}\sum_{II'}\langle\langle\Big(\frac{763}{512}P^{-2}_{II'}\operatorname{Tr}P^{-2}(\,\cdot\,) - \frac{13}{16}P^{-4}_{II'}(\,\cdot\,)\Big)\widetilde{s}^I_a(\,\cdot\,)\widetilde{s}^{I'}_{a'}(\,\cdot\,)\rangle\rangle,\\
S^{(0)}_{(a)(a'b')} &= \sum_{II'}\langle\langle\sqrt{\det P}P^{-2}_{II'}(\,\cdot\,)\widetilde{s}^I_a(\,\cdot\,)\widetilde{s}^{I'}_{a'b'}(\,\cdot\,)\rangle\rangle,
\end{aligned}$$

and analogously

$$\begin{aligned}
B_0^{(ab)(a'a')} &= \sum_{II'}\langle\langle\frac{\sqrt{\det P}}{P}^{-2}_{II'}(\,\cdot\,)b^{ab}_{\widetilde{\alpha}_I}(\,\cdot\,)b^{a'b'}_{\widetilde{\alpha}_{I'}}(\,\cdot\,)\rangle\rangle,\\
B_1^{(ab)(a'b')} &= \frac{\ell_P^4}{t}\sum_{II'}\langle\langle\frac{1}{V_v}\Big(\frac{763}{512}P^{-2}_{II'}\operatorname{Tr}P^{-2}(\,\cdot\,) - \frac{13}{16}P^{-4}_{II'}(\,\cdot\,)\Big)b^{ab}_{\widetilde{\alpha}_I}(\,\cdot\,)b^{a'b'}_{\widetilde{\alpha}_{I'}}(\,\cdot\,))\rangle\rangle,\\
B_0^{(ab)(a'b'c')} &= \sum_{II'}\langle\langle\frac{\sqrt{\det P}}{V_v}P^{-2}_{II'}(\,\cdot\,)b^{ab}_{\widetilde{\alpha}_I}(\,\cdot\,)b^{a'b'c'}_{\widetilde{\alpha}_{I'}}(\,\cdot\,)\rangle\rangle.
\end{aligned}$$

Now we can make the replacements (5.7) and obtain a Hamiltonian for the continuum fields $A_a(\vec{x})$, $E^a(\vec{x})$. A straightforward calculation yields the resulting equations of motion:

$$\begin{aligned}
\ddot{A}^d =&\partial_b\partial_{b'}A^{a'}S_{(d)(a)}\left(B^{(db)(a'b')} - B^{(a'b)(db')}\right)\\
&+\partial_b\partial_{b'}\partial^{c'}A^{a'}\Big[B^{(db)(a'b'c')} - B^{(a'b)(db'c')} + \left(S_{(d)(ac')} - S_{(a)(dc')}\right)\left(B^{(ab)(a'b')} - B^{(a'b)(ab')}\right)\Big]\\
&+\ldots
\end{aligned}\tag{5.8}$$

where we have used shorthands $S_{(a)(a')} \doteq S^{(0)}_{(a)(a')}+S^{(1)}_{(a)(a')}$ and similarly for the other tensors $S_{()()}, B^{()()}$.

Before we spell out the resulting dispersion relation, we use the rotation invariance of the graph on large scales: It is clear that

$$S_{(a)(a')} \sim \delta_{aa'}, \qquad S_{(a)(a'b')} \sim \epsilon_{aa'b'}.$$

For the tensors of higher rank, the situation is slightly more complicated: For rank four, the space of invariant tensors is three dimensional, the space of rank five tensors is ten dimensional. But if we take into consideration the symmetries of the terms, these tensors get contracted with, there is only one invariant tensor left in each case. We define

$$c_1^{(0/1)} \doteq \frac{1}{3}\sum_i S^{(0/1)}_{(a)(a)}, \qquad\qquad c_3^{(0/1)} \doteq \frac{1}{6}\left(\sum_{ab}B_{0/1}^{(ab)(ab)} - \sum_a B_{0/1}^{(aa)(aa)}\right),$$

$$c_2 \doteq \frac{1}{6}\sum_{abc} \epsilon^{abc} S^{(0)}_{(a)(bc)}, \qquad c_5 \doteq \frac{1}{6}\sum_{abc} \epsilon_{bac} B_0^{(bc)(acc)}.$$

A straightforward calculation shows that the equations of motion (5.8) simplify to

$$\ddot{\vec{A}}(t,\vec{x}) = \left(c_1^{(0)}c_3^{(0)} + c_1^{(0)}c_3^{(1)} + c_1^{(1)}c_3^{(0)}\right)\Delta\vec{A}(t,\vec{x}) + \left(c_2 c_3^{(0)} - c_1^{(0)}c_5\right)\Delta \operatorname{rot}\vec{A}(t,\vec{x}). \tag{5.9}$$

Note that in the last equation we have just kept terms of leading order and first order corrections, in the sense that we have explained in the previous section. Also we have eliminated a term containing $\operatorname{div} A$ by choosing the appropriate gauge.

Equation (5.9) leads to a chiral modification of the dispersion relation for electromagnetic waves: Let a unit vector $\vec{e}_3$ be given and choose $\vec{e}_1$,$\vec{e}_2$ such that the $\vec{e}_i$ form a righthanded orthonormal triple. Then a circularly polarized wave of helicity $\pm$, propagating in the direction given by e_3 can be written as

$$\vec{A}_k(t,\vec{x}) = A_0\left[\vec{e}_1\cos\left(\omega_\pm(k)t - k\,\vec{e}_3\cdot\vec{x}\right) \pm \vec{e}_2 \sin\left(\omega_\pm(k)t - k\,\vec{e}_3\cdot\vec{x}\right)\right].$$

This is a solution to the wave equation (5.9) provided that

$$\omega_\pm(k) = |k|\sqrt{\left(c_1^{(0)}c_3^{(0)} + c_1^{(0)}c_3^{(1)} + c_1^{(1)}c_3^{(0)}\right) \pm \left(c_2 c_3^{(0)} - c_1^{(0)}c_5\right)k}. \tag{5.10}$$

Thus we have found a chiral modification to the dispersion relation. Note that this chiral modification is similar but not completely analogous to the birefringence occurring for light propagation in some crystals. The latter effect is not isotropic, it also depends on the direction of propagation relative to the symmetry axes of the crystal, whereas the chiral effect found here is isotropic. This can be seen from the fact that nothing in the above formulae depends on the direction of the vector $\vec{e}_3$. We can now proceed to a discussion of results.

5.3 Discussion

Let us start the discussion of the results of the last section by considering the physical units and orders of magnitude of the various terms appearing. We will use F_{der}, the derivative term in the scalar field Hamiltonian, as an example – similar considerations apply to the other terms.

The classical term corresponding to $B_0^{(a)(a')} + B_1^{(a)(a')}$ is $\sqrt{\det q}\, q^{aa'}$. The latter is dimensionless, since q is. $B_0^{(a)(a')}$ has the structure

$$B_0^{(\cdot)(\cdot)} \sim \frac{1}{\text{Vol}}\frac{P^2}{\sqrt{\det P}} bb \tag{5.11}$$

where Vol is a volume. Since $[P] = \text{meter}^2$, $[P^2/\sqrt{\det P}] = \text{meter}$. b is also a length, unit-wise, so $B_0^{(a)(a')}$ is indeed dimensionless. $B_1^{(a)(a')}$ has the structure

$$B_1^{(\cdot)(\cdot)} \sim \frac{\ell_P^4}{t\text{Vol}} \frac{P^2 \operatorname{Tr} P^{-2} - 1}{\sqrt{\det P}} bb, \tag{5.12}$$

so it is again dimensionless as it should be. The structure of $B_0^{(ab)(a'b')}$ is

$$B_0^{(\cdot\,\cdot)(\cdot\,\cdot)} \sim \frac{1}{\mathrm{Vol}} \frac{P^2}{\sqrt{\det P}} bbbb, \tag{5.13}$$

so its unit is meter^2 which is the correct one for a term proportional to $|k|^4$ in the dispersion relation. As for orders of magnitude, we remark the following. Assume $q_{ab} = O(1)$ in the chosen coordinate system. Then

$$P = O(\epsilon^2), \qquad \mathrm{Vol} = O(\epsilon^3) \quad \text{and} \quad b = O(\epsilon). \tag{5.14}$$

Using (5.11) it follows that $B_0^{(a)(a')} = O(1)$, so the leading order term has the right order of magnitude. As for the order of magnitude of $B_1^{(a)(a')}$, we use (5.12) and (5.14) to conclude that

$$B_1^{(\cdot)(\cdot)} = O\left(\frac{1}{t}\frac{\ell_P^4}{\epsilon^4}\right) = O\left(\left(\frac{\ell_P}{L}\right)^{2-4\alpha}\right) = O(t^{1-2\alpha})$$

which is very small since $\alpha < 1/2$.
Consider finally $B_0^{(ab)(a'b')}$: From (5.13) and (5.14) we see that $B_0^{(ab)(a'b')} = O(\epsilon^2)$.
As for the other terms in the dispersion relation, similar results can be seen to hold: The leading order term has same unit and order of magnitude as the corresponding classical term and the ratio of leading order to first order correction is of order $t^{1-2\alpha}$.

We will now discuss the structure of the dispersion relations (5.4), (5.10). The coefficients appearing are given as graph averages of certain local geometric quantities of the random graph. Let us call these graph averages *moments* of the random graph prescription (RGP for short). So, in order to get numerical statements from the results of the last section, one has to fix the scale L, an RGP and compute the relevant moments. Such a computation might be hard to perform analytically, but a computer can easily determine the moments occurring in (5.4),(5.10) for a given RGP, so this calculation does not present a principal difficulty.

The more serious issue here is that there are certainly many RGPs, all leading to different graph averages and hence different predictions, and it is a priori not clear how one can single out the "right" one. We note however that for not too pathological RGPs, the graph averages should be approximately equal so that at least the size of the different terms in the dispersion relations is not too sensitive on the choice of the RGP. Moreover, again for a not too pathological RGP, the moments showing up in the dispersion relations should be related. To give an example, a plausible assumption is that

$$\langle\langle\sqrt{\det P}\rangle\rangle \approx \left(\langle\langle\frac{1}{\sqrt{\det P}}\rangle\rangle\right)^{-1}$$

and that their difference would not depend very strongly on the chosen prescription. Thus there will be approximate relations between the different coefficients in the dispersion relations which are not affected by the choice of a specific RGP.

Moreover we note that the leading order terms in the coefficients depend on the RGP. This might at first seem to be a problem as well, since it means that we will have to tune the RGP in such a way that the leading order terms assume their classical values. On the other hand, this can be seen as a blessing: Fixing the leading order term means to fix one moment of the RGP. Via the relations conjectured above, this will also approximately fix other moments, independently of the specific RGP assumed, and thereby to a certain extent the higher order corrections.

Investigations in this direction are worthwhile but beyond the scope of the present work. Let us for the rest of this section assume that a prescription is fixed and the relevant graph averages have been computed.

Next we observe that two different sorts of corrections appear in the dispersion relations: The first sort of correction is simply a correction to the leading order term coming out of the fluctuation calculation of section 4. Its relative magnitude was found to be $t^{1-2\alpha}$. We will call this sort of correction a *fluctuation correction*.

The other sort of correction is a term containing a higher power of $|k|$ as compared to the standard dispersion relation. We will call this kind of correction a *lattice correction*. We have demonstrated for the example of $B_0^{(ab)(a'b')}$ that the terms proportional to $|k|^4$ are of the order ϵ^2, therefore the relative magnitude of the lattice corrections is of the order

$$O\left(\frac{\epsilon^2}{\lambda^2}\right) = \frac{L^2}{\lambda^2}\ O\left(t^\alpha\right) \leq O(t^\alpha).$$

Similarly the terms proportional to $|k|^3$ in the dispersion relation for the electromagnetic field are of the order $t^\alpha L/\lambda$.

When comparing our results for the electromagnetic field with the ones of [?, ?] we find the following: The result of Pullin and Gambini [?] does not contain any fluctuation corrections. This is however not result of the calculation but rather assumed from the beginning. As for the lattice corrections, they find a chiral modification to the dispersion relation as we do here. The relative magnitude of the correction is however ℓ_P/λ.

Alfaro et al. [?] also do not have fluctuation corrections by assumption. They find the helicity dependent correction of [?] and the present work, again of the order ℓ_P/λ. They also get higher order corrections the precise structure of which depends on a parameter which is not fixed.

Thus our results agree with that of [?, ?] as far as the structure of the dispersion relation is concerned. We additionally have fluctuation corrections and, most importantly, the corrections found *do not scale with an integer power of ℓ_P, contrary to their finding.* This signals a warning to assumptions made in [?] to take into account only corrections which are of the order $(\ell_P/L)^n$ where n is an integer. Notice that the fluctuation correction and the lattice correction are equal at $\alpha = \frac{1}{3}$. Thus the leading correction is always of the order of at least $t^{1/3}$.

Finally we should make a few remarks concerning a possible detection of the corrections in experiments. The fluctuation corrections will not show up in an experiment testing for a frequency dependence of the velocity c of light, since they merely correspond to a frequency independent shift of c. Also, these corrections are certainly not measurable by measuring the flight-time of photons since their velocity would already be the "bare" leading order term plus the fluctuation correction. Fluctuation corrections may however be measurable by comparing flight-times of photons in different geometries, since the corrections will change when the calculations presented in this chapter are repeated with LQC approximating a non-flat spacetime. To discuss how this could be done in practice is however beyond the scope of the present work.

Whether the lattice corrections are big enough to be detectable in the data from current or planned γ-ray burst observations crucially depends on the values of α and L. For the value $\alpha = 1/3$ which renders fluctuation and lattice corrections equal in magnitude (and which is close to the lower bound value 2/5 derived in [?]), and L of the order of a γ-ray wavelength, a rough estimate shows that *the lattice corrections would indeed be detectable in the foreseeable future.*

So, to conclude this chapter, we should repeat that not too deep a significance should be attached to the precise values of the coefficients in the dispersion relations obtained: There are still some ambiguities present in the GCS which we will discuss below: The quantization of the Hamiltonians, in the procedure to obtain the dispersion relations from the expectation values and, as a consequence, in the coefficients themselves. Also the replacement SU(2) $\rightarrow$ U(1) will certainly affect the precise numerical outcome. Most significantly, so far we have little control on what will happen to the size of quantum corrections when our kinematical coherent states are replaced by physical ones. Within our kinematical scheme the structure of the corrections, as well as the orders of magnitude t^{α}, $t^{1-2\alpha}$ of the two sorts of corrections are robust, however. Thus we are possibly in trouble because such corrections seem to lie in the detectable regime. If such corrections are not found, then presumably it is not justified to use kinematical coherent states.

Finally, the approximate relations between the different graph averages will make the predictions of a more complete calculation much less dependent on the random graph prescription chosen, then one might at first fear. Similar remarks apply if, as advocated for example in [?], instead of working with a fixed random graph, one averages over many of them. (Notice that also in that case, averaging procedures are not unique). In order to remove those ambiguities one should probably set up a variational principle in order to optimize a family of semiclassical states according to a given set of observables.

6 Summary and Outlook

In the this work we have presented a calculation of dispersion relations for the scalar and the electromagnetic field coupled to quantum general relativity. These dispersion relations bear corrections to the standard ones, due to the discreteness of the states of the geometry and to the bound on the uncertainty product of configuration and momentum variables in QGR. The calculations rest on the quantization of the matter parts of the Hamilton constraint given in [?] and the coherent states for QGR constructed and analyzed in [?, ?, ?, ?]

Corrections to dispersion relations due to QGR were also computed in [?, ?] and the present work partly rests on the ideas implicit and explicit in these pioneering works. The form of the correction term in the dispersion relation for the electromagnetic field found in the present work agrees with that of [?, ?]. This is not too big a surprise since there is no other rotation invariant term in $\vec{k}$ of the same order. However, we find important differences in the order of magnitude of the effects, as compared to [?, ?]. Moreover, the results of the present work are more specific, since a definite class of semiclassical states, the coherent states for QGR are employed in the calculation.

Rather than making precise numerical predictions, the aim of the present work is to demonstrate the steps necessary in such a calculation, to highlight the issues that remain to be clarified and to give a robust estimate of the size of the effects.

In this spirit, we have simplified the calculation of the expectation values in 4 by replacing the full gauge group by its Iönü-Wigner limit $\mathrm{U}(1)^3$. This replacement will certainly affect the precise numerical outcome but not the order of magnitude of the correction. Also we have not specified a prescription for obtaining random graphs, but only assumed general properties that such a procedure will have. Most importantly, the effect of using kinematical rather than physical coherent states is presently not well understood.

The main achievements of the present work can be summarized as follows:

The calculation given in section 4 shows how expectation values of complicated operators in coherent states for quantum general relativity can be computed and there is no principal difficulty in repeating such a calculation for the full gauge group SU(2).

Perhaps even more important are the order of magnitude estimates of the resulting effects obtained in this work: They depend on very few parameters and will continue to hold true when more general complexifier coherent states [**?**] are used. The main choices that enter are:

- A complexifier C has to be chosen for the construction of the coherent states. (Of course, there are more general semiclassical states than coherent ones).
- A class of observables has to be chosen that should be approximated well by the coherent states.
- A (random, averaged) graph has to be chosen.

The other parameters are fixed by the above choices: The requirement that $C/\hbar$ is dimensionless forces the parameter t in the definition of the resulting coherent states to be $(\ell_P/a)^n$ where n is some positive number and a a length scale which is not yet fixed.

The nature of these observables (do they involve one-, two- or three-dimensional integrations? etc.) determines a) a length scale L and b) the exponent β in the expression for the classical error $(\epsilon/L)^{2\beta}$.

The length scale a gets fixed to be L by requiring fluctuations of configuration and momentum degrees of freedom to be equal. Finally, the typical edge-length ϵ of the random graph is found to be a weighted geometric mean by requiring the fluctuations to be minimal. Thus, at least within the vast class of complexifier coherent states, the structure of the ambiguities and their principal effects on the orders of the magnitudes of the quantum corrections *can be neatly classified*!

Many things remain to be done before one can really obtain reliable predictions of observable effects from quantum general relativity:

The procedure used to obtain dispersion relations from the discrete classical Hamiltonians has to be further analyzed, and rigorously justified at least in models which can be solved analytically. The influence of the choice of a random graph should be investigated, and concrete procedures have to be implemented. A more distant goal is to also analyze possible back reaction effects of the matter on the gravitational field. These were neglected in [**?**] and in this work since it would require to solve the combined matter – geometry Hamiltonian constraint and force us to work with physical coherent states.

Thus, although we certainly did not carry out a first principle calculation, we hope to have made a modest contribution to an understanding what the principal problems are and how such a computation could possibly be carried out in principle. Also, we hope to have demonstrated that QGR is still far from making reliable semiclassical predictions until one is convinced of the physical relevance of a definite scheme. However, it should have become clear that once such a scheme has been identified, QGR *is* able to provide *precise* numerical predictions. In any case, at least for the limited purpose of showing that some version of the quantum Hamiltonian constraint is correct (for which kinematical coherent states are unavoidable), the results of the present two papers should be relevant.

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